

Spontaneous imbibition in randomly arranged interacting capillaries

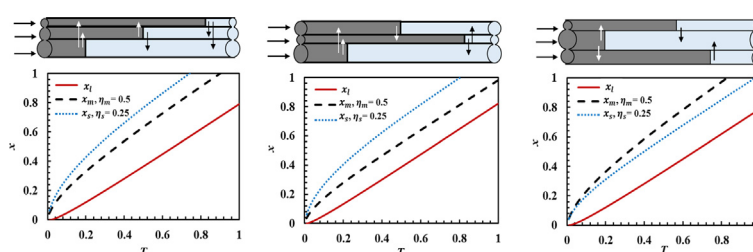
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HIGHLIGHTS

- VOF simulations used to describe imbibition in an interacting two capillary system.
- Developed 1-D model for interacting capillaries using inferences from VOF simulations.
- 1-D model extended for three capillary system for random arrangement of capillaries.
- Imbibition depends on radii and arrangement of capillaries in three capillary system.

GRAPHICAL ABSTRACT



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ABSTRACT

Spontaneous imbibition in a porous medium is known to have two macroscopic fronts, the leading one in narrow pores and the lagging front in the wider pores. This behavior contradicts the behavior predicted by Washburn law due to interaction of the pores in the porous medium. In this work, we investigate the spontaneous imbibition behavior in a randomly arranged interacting capillaries system and propose a one dimensional Washburn like model which can be used to upscale the porous medium properties. We first use Volume of Fluid (VOF) simulations, which tracks the fluid-fluid interface in the multiphase flow, and show several approximations that can be used to develop a quasi one-dimensional Washburn like model for interacting capillaries. Using these approximations, we build a model for two and three capillary systems, which is able to mimic the spontaneous imbibition in a three-dimensional interacting capillary system simulated using VOF. The one-dimensional model for three capillaries shows a strong dependence of the spontaneous imbibition behavior on the arrangement of the interacting capillaries and their relative radii. We show using the one-dimensional model, that the meniscus does not always lead in the smallest radius capillary in the interacting system. We also show that the spontaneous imbibition in the whole system also deviates from the Washburn like diffusive behavior.

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1. Introduction

Capillary driven flow is important for many scientific and engineering applications, for example recovery of oil from the porous matrix of a fractured reservoir (Tanino and Blunt, 2012; Zhang et al., 2014), geological CO₂ sequestration (Elenius and Gasda, 2013; Celia et al., 2015), contaminant transport in aquifers

(Birdsell et al., 2015; Xiao et al., 2012; Shou et al., 2010), design of textile fabrics (Sarkar et al., 2009; Du et al., 2008) and microfluidic devices (Cate et al., 2014; Gunda et al., 2011). The foundation of the capillary driven flow was laid by Lucas (1918) and Washburn (1921), who described that the fluid imbibition in an empty capillary follows diffusion like behavior, i.e., $l^2 = Dt$. Here, l is the length imbibed by the fluid in time t and D is the effective diffusivity coefficient dependent on the capillary and the fluid properties. More precisely, $D = \frac{\gamma r}{2\mu}$ for a capillary of radius r , being imbibed by a completely wetting fluid of viscosity μ and surface tension γ . Several

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modifications of the Lucas-Washburn law have been proposed. Hultmark et al. (2011) suggested that the length traveled by the wetting fluid deviates from the diffusive dynamics, if the capillary is initially filled with a viscous fluid. Budaraju et al. (2016), Gorce et al. (2016) and Reyssat et al. (2008) modified the Washburn's law for axially varying capillaries. Waghmare and Mitra (2012) studied the imbibition in closed rectangular micro-channels, and Ouali et al. (2013) described the imbibition in non-circular open channels.

A porous medium is made up of several pores, which are non-uniform in size and shape. Therefore, imbibition dynamics in a bundle of independent capillaries of varying shapes and sizes have been used to describe the flow in porous media. Purcell (1949) used the capillary bundle model to study the capillary pressure curves in a porous medium, which were later used by Brooks and Corey to find the relative permeability for oil and gas reservoirs (Corey, 1954; Brooks and Corey, 1964). Using bundle of tubes model, Dahle et al. (2005) described the dynamic effects on capillary pressure-saturation relationship and Bartley and Ruth (1999, 2001) explained relative permeability of a porous medium. The capillaries collectively represent an ideal porous medium, which can explain the underlying dynamics of imbibition in detail. In bundle of capillaries, the meniscus in the largest radius capillary leads according to the Washburn's diffusive law. This is because, the driving force of the capillary pressure scales as $\frac{1}{r}$ and viscous resistance scales as $\frac{1}{r^2}$ for a capillary of radius r .

In the above studies, the representation of a porous medium by a bundle of capillaries ignores the hydrodynamic interaction of the pores found in naturally occurring porous media. Bico and Qu  r   (2003) and Dong et al. (1998) explained that the hydrodynamic interaction between the capillaries alters the imbibition dynamics significantly. In hydrodynamically interacting capillaries, the meniscus in the small radius capillary leads, contrary to the Washburn's law (Dong et al., 2005, 2006; Ashraf et al., 2017; Li et al., 2017). This behavior was seen experimentally in non-circular capillaries by Unsal et al. (2009). To develop a Washburn like one dimensional model for interacting capillaries, Dong et al. (1998) assumed that the cross-flow between the capillaries is negligible. They also assumed that in the section where capillaries are filled with the same phase, the pressure gradient in both the capillaries is same. However, their model is valid for capillaries which are arranged in the order of their radii. Ashraf et al. (2017), by analogy developed a flow model for parallel layers of packed beads assuming that pressure in the two layers is same whenever same fluid is present in both the layers, while the fluid transfer between the layers is assumed to be significant which occurs at the imbibing fluid front. However, their model neglects the viscosity of the resident fluid. In the above models, the validity of the assumptions made is still not known.

In the present work, we use Computational Fluid Dynamics (CFD) to investigate capillary driven flow in interacting capillaries. We use Volume of Fluid (VOF) method, which tracks the fluid-fluid interface, to describe the dynamics of imbibition in a hydrodynamically interacting two capillary system. Using the high fidelity simulations, we present the inferences that can be used to build a quasi one-dimensional lubrication approximation model for interacting capillaries. We develop the quasi one-dimensional model based on the inferences and then compare with the VOF simulations. The one-dimensional model sufficiently explains the invasion dynamics in a two capillary system. We further use the inferences from the VOF simulations to develop spontaneous imbibition model in an interacting three capillary system for any random arrangement of capillaries. Using the three capillary system, we show that the flow behavior is strongly dependent on the radii and the relative arrangement of the capillaries. The new model can

be used to study the imbibition in porous media (Sun et al., 2016; Mullins et al., 2007).

2. VOF model for capillary driven flow

Volume of Fluid (VOF) method was introduced by Hirt and Nichols (1981) for multiphase flow. The method solves volume-averaged Navier-Stokes equation for two fluid phases and captures the fluid-fluid interface by tracking the volume fraction of a phase in a grid block. For the capillary driven flow, a capillary initially filled with a non-wetting fluid of viscosity μ_{nw} and density ρ_{nw} is considered. Then, one end of the capillary is brought in contact with a completely wetting fluid of viscosity μ_w and density ρ_w . The contact angle of the wetting phase with the capillary surface is θ in presence of the non-wetting phase. The opposite end of the capillary is assumed to be in contact with the non-wetting phase at all times, so that no capillary end effects are present at the outlet boundary. To model this phenomenon using VOF, the Navier-Stokes equation and the equation of continuity as shown in Eqs. (1) and (2) are solved for velocity and pressure of the volume averaged mixture of fluids.

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla \cdot (\rho \mathbf{v} \mathbf{v}) = -\nabla P + \nabla \cdot [\mu(\nabla \mathbf{v} + (\nabla \mathbf{v})^T)] + \mathbf{F}, \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{v} = 0. \quad (2)$$

Here, $\mathbf{v}$ is the velocity vector, ρ is the density, μ is the viscosity of the mixture and P is the pressure. The term $\mathbf{F}$ in Eq. (1) is the volumetric force due to capillary pressure, which is a result of the surface adhesion forces. In this study, we have considered horizontal capillaries where the effect of gravity will be negligible (Das and Mitra, 2013). All the grid-cells are occupied by either the wetting phase or the non-wetting phase except for the interface cells. The density and the viscosity for the interface cells are defined by,

$$\rho = (1 - \alpha)\rho_{nw} + \alpha\rho_w, \quad (3)$$

$$\mu = (1 - \alpha)\mu_{nw} + \alpha\mu_w, \quad (4)$$

where α is the volume fraction of the wetting phase. The unknown α can be solved by phase continuity equation,

$$\frac{\partial}{\partial t}(\alpha\rho_w) + \nabla \cdot (\alpha\rho_w \mathbf{v}) = 0, \quad (5)$$

The volumetric capillary force $\mathbf{F}$ in Eq. (1) is a function of the curvature of the fluid-fluid interface in the capillary. Continuum Surface Force (CSF) method by Brackbill et al. (1992) is used which defines volumetric force $\mathbf{F}$ as

$$\mathbf{F} = -\gamma\kappa\mathbf{n}, \quad (6)$$

where γ is the interfacial tension between the wetting and the non-wetting phases, $\mathbf{n} = \nabla\alpha$ is the vector normal to the interface and κ is the curvature of the interface which is defined as,

$$\kappa = \frac{1}{|\mathbf{n}|} \left[\left(\frac{\mathbf{n}}{|\mathbf{n}|} \cdot \nabla \right) |\mathbf{n}| - \nabla \cdot \mathbf{n} \right]. \quad (7)$$

At the wall boundary, the CSF method calculates the unit normal vector $\hat{\mathbf{n}}$ by,

$$\hat{\mathbf{n}} = \hat{\mathbf{n}}_1 \cos \theta + \hat{\mathbf{n}}_2 \sin \theta, \quad (8)$$

where θ is the contact angle, $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are unit vectors normal and tangential to the wall, respectively.

VOF method is known to induce spurious currents in surface tension driven flows because the force due to surface tension as mentioned in Eq. (6) is obtained by calculating the curvature and

the vector normal to the interface (Pan et al., 2015; Nekouei and Vanapalli, 2017; Harvie et al., 2006). The curvature and the vector normal are defined as a function of the fluid volume fraction while they are properties of the interface. This sometimes results in an erroneous direction of the surface tension force causing spurious currents. Spurious currents disrupt the interface when the difference between the density and viscosity of the two fluids is high. However, spurious currents can be eliminated by choosing suitable viscosity, density and courant number (Pan et al., 2015). In our simulations, we consider equal viscosity and density of the imbibing and residing phases and by using a suitable courant number, the spurious currents are eliminated.

For the geometry used in this study, to initiate the imbibition and to keep the inlet boundary in contact with the wetting fluid, the volume fraction of the wetting fluid at the boundary cells is defined unity. We have used ANSYS Fluent®, 15.0 to solve Eqs. (1), (2) and (5). These equations can be solved simultaneously for α , P , $\mathbf{v}$; however, the procedure becomes computationally expensive. Therefore, we have solved Eq. (5) first explicitly and the value of α at the new time step is then used in Eqs. (1) and (2) to solve for P and $\mathbf{v}$. The coupling of P and $\mathbf{v}$ is carried out using PISO (Pressure Implicit with Splitting Operators) algorithm by Ubbink and Issa (1999). A first order upwind scheme and Non-Iterative Time Advancement Scheme (NITA) available in Fluent are used to implicitly solve the unsteady state equations. To reconstruct the fluid-fluid interface, a piecewise linear approach is used, called as the Geo-reconstruct. The geometric construction and meshing of capillaries is done in ANSYS ICEM CFD.

3. Validation of the VOF model

To validate the numerical model of the capillary driven flow, we consider two capillaries of radii $r_l = 1$ mm and $r_s = 0.5$ mm, and length 100 mm. We carry out simulations in coarse and fine grids for the two capillaries to check for grid independence and then compare with the one-dimensional model given by Budaraju et al. (2016). We discretize the 1 mm radius capillary in 155×388 grids in cross sectional $\times$ axial directions for coarse meshing and 225×531 grids for fine meshing. Similarly, we discretize 0.5 mm radius capillary in 66×690 and 121×739 grid cells. The capillaries are initially filled with a fluid of viscosity $\mu_{nw} = 0.01$ Pa·s and density $\rho_{nw} = 998$ kg·m⁻³. One end of the capillaries is brought in contact with a wetting fluid of viscosity $\mu_w = 0.01$ Pa·s and density $\rho_w = 998$ kg·m⁻³ at time $t = 0$. The interfacial tension between the two fluids is $\gamma = 0.0206$ N·m⁻¹. The imbibing fluid completely wets the surface of the capillary in presence of the resident fluid, i.e., the contact angle of the imbibing fluid $\theta = 0^\circ$. Fig. 1 shows the interface position with time for the two capillaries, for the two discretizations considered. We see in Fig. 1 that, the meniscus in the 1 mm radius capillary travels faster as mentioned earlier. Refining the grid size does not change the solution, therefore, coarse grid is sufficient to solve the capillary driven flow.

In Fig. 2, we compare the meniscus displacement in the capillaries using VOF simulations and the lubrication approximation model given by Hultmark et al. (2011) for a capillary driven flow of a viscous fluid displacing another viscous fluid. The length traveled by the meniscus, l , in time t using Hultmark et al. (2011) is given by,

$$(\mu_w - \mu_{nw})l^2 + 2\mu_{nw}Ll = \frac{c\gamma r}{4}t, \quad (9)$$

where L is the total length of the capillary, r is the radius and $c = 2 \cos \theta$. In the simulations, we have considered $\mu_{nw} = \mu_w = \mu$, which reduces Eq. (9) to, $l = \frac{\cos \theta \gamma}{4\mu L}rt$. We see that the simulation

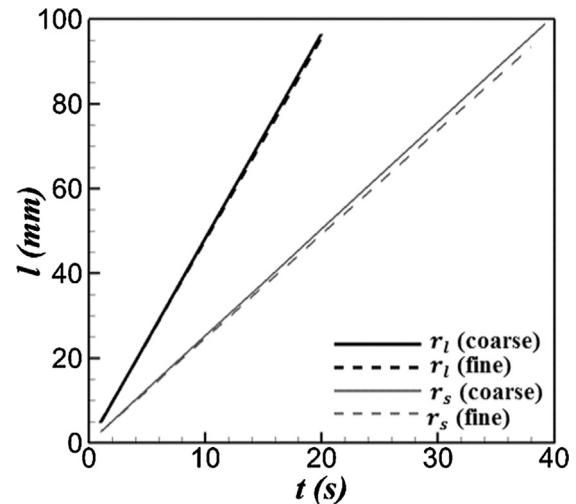


Fig. 1. Spontaneous imbibition in independent capillaries of radii $r_l = 1$ mm and $r_s = 0.5$ mm. Imbibed length vs time for fine grid of 225×531 cells and coarse grid of 155×388 cells in capillary of radius r_l ; the capillary of radius r_s is meshed with fine grid of 121×739 cells and coarse grid of 66×690 cells.

results have an excellent agreement with the model and hence we call them high fidelity simulation. The meniscus formation time recorded from the simulations and the inertial regime time $\frac{\rho r^2}{4\mu}$ (Quérré, 1997; Das et al., 2012), together are of the order 0.1 s and 0.5 s for capillaries of radii 0.5 mm and 1 mm, respectively. These times are considered negligible as the total imbibition time is 19.5 s and 39.5 s for the 1 mm and the 0.5 mm radii capillaries, respectively.

Another validation criterion that we have used is the pressure jump at the meniscus. In Fig. 2(b), we show the pressure along the length of the capillaries at 10 s. The 0.5 mm capillary shows a capillary pressure jump of 78.1 Pa and the 1 mm capillary shows a capillary pressure jump of 37.8 Pa at their respective menisci at 10 s. The theoretical capillary pressure using the Young-Laplace equation for the 0.5 mm capillary is 82.4 Pa and for the 1 mm capillary is 41.2 Pa (Gauss, 1877). This shows a good agreement between the theoretical and simulation pressures.

4. VOF simulation of spontaneous imbibition in interacting capillaries

In this section, we investigate the spontaneous imbibition in hydrodynamically interacting capillaries. The radii of the capillaries are 1 mm, referred as the large capillary and 0.5 mm, referred as the small capillary. The capillaries interact along the length and the interaction depends on the distance between their centers as shown in the insets of Fig. 3. Four values of center-to-center distance are considered in this study, i.e., 1.495 mm, 1.44 mm, 1.35 mm and 1.25 mm as shown in Fig. 3. Each of the capillary systems are discretized in about 300,000–400,000 cells depending on the geometry. The average volume of the discretized cell is 1.21×10^{-3} mm³, which is finer than the average size of the coarse grid considered in Section 3. The VOF simulation of a wetting fluid displacing a resident non-wetting fluid in the capillary system proceeds as before.

Fig. 3 shows the axial pressure vs. distance graph at 10 s in all the four interactions considered in the respective inset figures. It can be observed that the meniscus in the small radius capillary is leading contrary to the observations in the non-interacting capillaries in Section 3. It should be noted from Fig. 3 that the pressure in both the capillaries is same before the ‘first meniscus’ in the

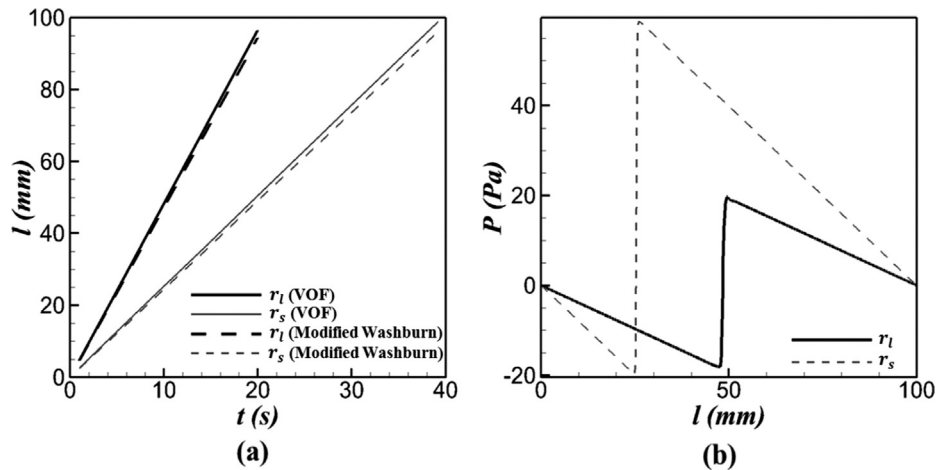


Fig. 2. Spontaneous imbibition in independent capillaries of radii $r_l = 1$ mm and $r_s = 0.5$ mm. (a) Comparison of VOF simulations and theoretical results from modified Washburn's law for distance vs time. (b) Pressure vs distance plot at 10 s.

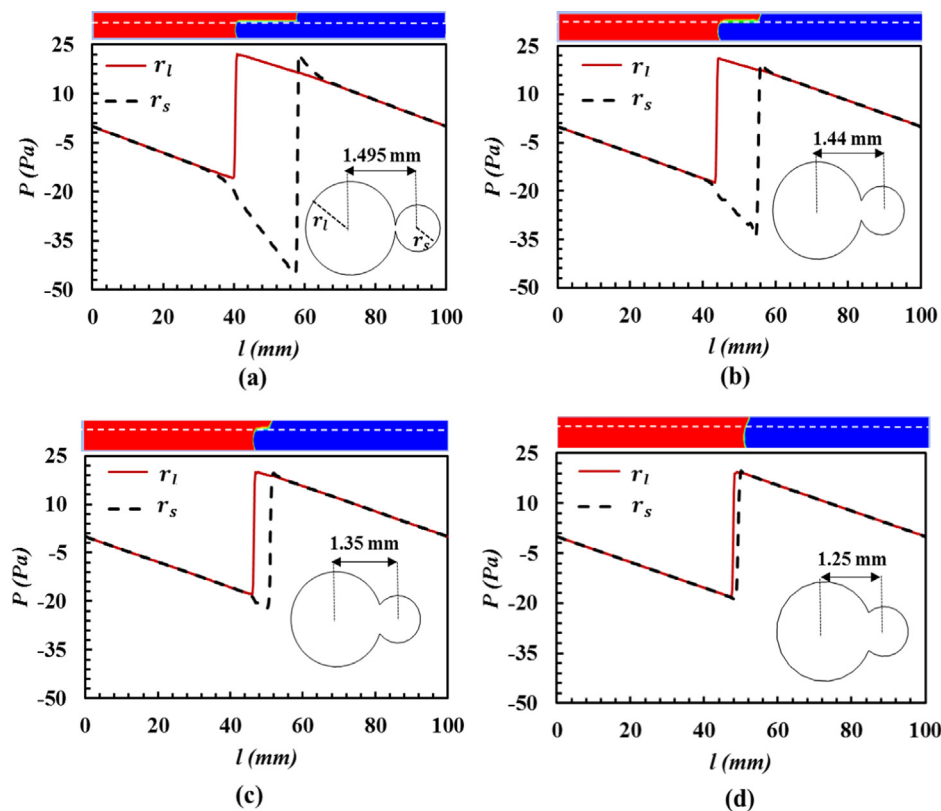


Fig. 3. Axial pressure vs imbibition length plot at 10 s for spontaneous imbibition in interacting capillaries of radii $r_l = 1$ mm, $r_s = 0.5$ mm with a center-to-center distance of (a) 1.495 mm, (b) 1.44 mm, (c) 1.35 mm and (d) 1.25 mm. The visuals from VOF simulations at 10 s can be observed above each of the pressure gradient graphs for respective center-to-center distances. The insets shown in the pressure gradient graphs are the cross sections of the respective interacting capillary systems.

large capillary and after the ‘second meniscus’ in the small capillary. This concurs with the assumption made in studies by Dong et al. (1998) and Ashraf et al. (2017) that, the pressure and pressure gradient in both the capillaries will be same if the same fluid is present. Moreover, as both the wetting and the non-wetting fluids have the same viscosity, the pressure gradient before the first meniscus is same as the pressure gradient after the second meniscus.

The change in the capillary cross-section from a circle to a segment changes the curvature of the menisci in both the capillaries as shown in Fig. 4 by iso-surfaces of $\alpha = 0.5$. The curvature is less

for more interaction leading to a lower capillary pressure. The curvature of the larger of the two interacting capillaries does not change significantly as shown by the menisci in Fig. 4 and the capillary pressure in Fig. 3. Closer the capillary pressure in the two capillaries, the closer are the two menisci as shown in Fig. 3.

Fig. 5 shows the streamlines in the interacting capillary system, where the red color shows the wetting fluid and the blue color shows the non-wetting fluid. A close observation of the streamlines near the menisci shows that the wetting fluid transfers from the large capillary to the small capillary just before the first meniscus as shown in Fig. 5(b). Similarly, the non-wetting fluid transfers

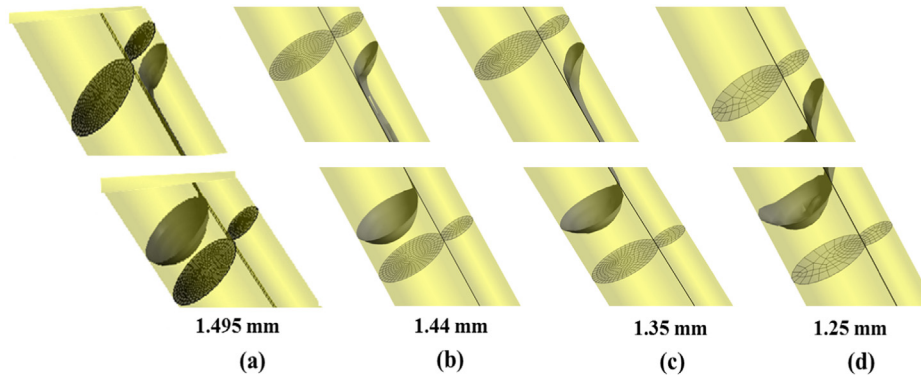


Fig. 4. Menisci shape at 10 s for spontaneous imbibition in interacting capillary systems having radii $r_l = 1$ mm, $r_s = 0.5$ mm and a center-to-center distance of (a) 1.495 mm, (b) 1.44 mm, (c) 1.35 mm and (d) 1.25 mm, for volume fraction, $\alpha = 0.5$.

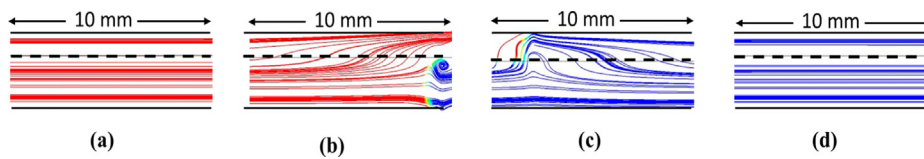


Fig. 5. Streamlines indicating the fluid transfer between the tubes at 10 s. (a) Shows that before the first meniscus there is no fluid transfer, (b) shows that the wetting fluid transfers from the large capillary to the small capillary before the first meniscus, (c) shows that the non-wetting fluid transfers from the small capillary to the large capillary after the second meniscus and (d) shows that after the second meniscus there is no fluid transfer between the tubes. Each of the sections shown in (a), (b), (c), (d) are 10 mm in length, while the total length of the interacting capillary system is 100 mm.

from the small capillary to the large capillary just after the meniscus in the small capillary as shown in Fig. 5(c). Fig. 5(a) and (d) shows streamlines away from the menisci, showing no fluid transfer. If we closely observe the axial pressure near the menisci from Fig. 3, the pressure in the capillaries just before the meniscus in the large capillary and just after the meniscus in the small capillary are slightly different. Due to this pressure deviation in the wetting fluid just before the first meniscus, the wetting fluid flows from the large capillary to the small capillary. The fluid transfer occurs because, the fluid prefers to travel the path of least resistance. The resistance Ω offered by a capillary of radius r to a fluid of viscosity μ flowing through a distance l , with a flow rate of Q and a pressure drop of ΔP is given by Westerhof et al., as (Westerhof et al., 2010),

$$\Omega = \frac{\Delta P}{Q} = \frac{8\mu l}{\pi r^4}. \quad (10)$$

The resistance encountered in the small capillary is more than that in the large capillary for the same length of fluid flow. It is evident from Eq. (10) that, for a pressure drop equivalent to the capillary pressure of the large capillary experienced by the fluid till the first meniscus, the resistance in the large capillary is less than that in the small capillary.

We see from Fig. 3(a) that for a center-to-center distance of 1.495 mm, due to this fluid transfer, the pressure gradient in the small capillary increases from $0.4 \text{ Pa}\cdot\text{mm}^{-1}$ to $1.45 \text{ Pa}\cdot\text{mm}^{-1}$, i.e., the velocity of the wetting fluid in the small capillary increases to 3.62 times due to the wetting fluid transfer. However, the change in the fluid velocity in the large capillary due to the fluid transfer is not significant as seen in Fig. 3(a). This is because of the large cross-section area of the large capillary, that a small decrease in the flow rate translates to a very small decrease in the velocity and pressure gradient. Similar increase in the pressure gradients can be observed for the cases shown in Fig. 3(b), (c) and (d).

We infer from the above discussion that (a) pressure equilibrates between the two capillaries away from the menisci, if the

same fluid is present in both the capillaries. (b) With increase in interaction, the capillary pressure in the small capillary decreases, which is due to the change in curvature. (c) Fluid transfer is significant which occurs only near the menisci locations, and therefore, lubrication approximation can be used assuming fluid transfer only at the menisci location.

5. One-dimensional model for imbibition in interacting capillaries

We now develop a quasi one-dimensional model, using lubrication approximation, for spontaneous imbibition in interacting capillaries based on the observations from VOF simulations. We consider that two capillaries of radii r_l (large capillary) and r_s (small capillary) of length L are connected along the axes as shown in Fig. 6. The capillaries are initially filled with a non-wetting viscous fluid of density ρ_{nw} and viscosity μ_{nw} . At $t = 0$, the left end of the interacting capillary system is brought in contact with a wetting fluid of viscosity μ_w and density ρ_w . We consider pressure equilibration in the sections of the capillaries where the same fluid is present in both the capillaries and the fluid transfer occurs at the menisci locations only, as shown in Fig. 6(a) and (b). After time t from the start of imbibition, we assume that the meniscus in the large capillary is at a distance l_l from the inlet and the meniscus in the small capillary leads at a distance l_s from the inlet. The viscous pressure loss in the large capillary till l_l is given by Hagen-Poiseuille law as,

$$P_w(l_l, t) - P_o = -\frac{8\mu_w l_l}{r_l^2} U_l(t), \quad (11)$$

where P_o is the atmospheric pressure at the inlet, $U_l(t)$ is the average wetting phase velocity in the large capillary, and $P_w(l_l, t)$ is the pressure in the wetting phase at l_l in the large capillary, i.e., point A in Fig. 6(b). The pressure gradient till l_l is same in both the capillaries, which implies that $\frac{U_l(t)}{r_l^2} = \frac{U_s(t)}{r_s^2}$, where $U_s(t)$ is the average velocity

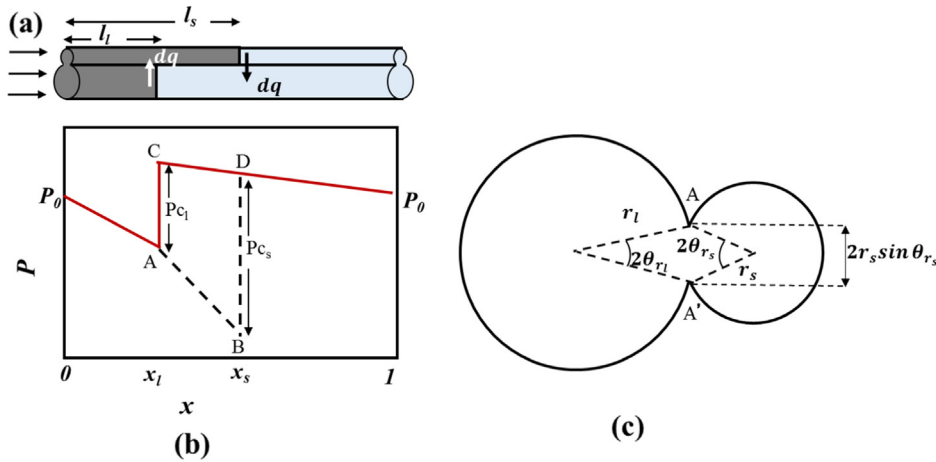


Fig. 6. An interacting two capillary system. (a) Schematic of fluid transfer at menisci locations, (b) schematic of pressure vs distance graph during imbibition, (c) the cross section of the capillaries used to calculate the capillary pressure in each of the capillaries.

of the wetting phase in the small capillary from inlet to the meniscus at l_l . We assume that the wetting fluid transfers from the large capillary to the small capillary, at the first meniscus with a flow rate of dq . The fluid transfers before the meniscus at l_l , therefore, the meniscus velocity in the large capillary is given by,

$$\frac{dl_l}{dt} = U_l(t) - \frac{dq}{\pi r_l^2}. \quad (12)$$

The velocity of the wetting phase in the small capillary after the fluid transfer becomes $U_s(t) + \frac{dq}{\pi r_s^2} = U_l(t) \frac{r_l^2}{r_s^2} + \frac{dq}{\pi r_s^2}$. The viscous loss in the large capillary due to non-wetting phase flowing from l_l to l_s is,

$$P_{nw}(l_s, t) - P_{nw}(l_l, t) = -\frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(U_l(t) - \frac{dq}{\pi r_l^2} \right), \quad (13)$$

where $P_{nw}(l_i, t)$ is the non-wetting phase pressure at position l_i ($i = l, s$). The viscous loss in the small capillary due to wetting phase flowing from l_l to l_s is,

$$P_w(l_s, t) - P_w(l_l, t) = -\frac{8\mu_w(l_s - l_l)}{r_s^2} \left(U_l(t) \frac{r_l^2}{r_s^2} + \frac{dq}{\pi r_s^2} \right), \quad (14)$$

where $P_w(l_i, t)$ is the wetting phase pressure at position l_i ($i = l, s$). Before l_s there is no fluid transfer, therefore we write the meniscus velocity in the small capillary as,

$$\frac{dl_s}{dt} = U_l(t) \frac{r_l^2}{r_s^2} + \frac{dq}{\pi r_s^2}. \quad (15)$$

After the meniscus in the small capillary we have non-wetting fluid in both the capillaries and the non-wetting fluid transfers from the small capillary to large capillary. The pressure gradient from l_s to L is same in both the capillaries. This is possible when average velocity in the small and the large capillaries are $U_s(t)$ and $U_l(t)$. The viscous pressure loss from l_s to L in the capillaries can be written as

$$P_o - P_{nw}(l_s, t) = -\frac{8\mu_{nw}(L - l_s)}{r_l^2} U_l(t). \quad (16)$$

The Young-Laplace pressure jumps at l_l and l_s are given by Eqs. (17) and (18), respectively as,

$$P_{nw}(l_l, t) - P_w(l_l, t) = P_{C1}, \quad (17)$$

$$P_{nw}(l_s, t) - P_w(l_s, t) = P_{Cs}. \quad (18)$$

Eliminating $P_{nw}(l_i, t)$, $P_w(l_i, t)$ and by replacing $U_l(t)$, $U_s(t)$, dq from Eqs. (11)–(18), we get governing equation for menisci position as,

$$P_{C1} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right), \quad (19)$$

$$P_{Cs} - P_{C1} = (l_s - l_l) \left(\frac{8\mu_w}{r_s^2} \frac{dl_s}{dt} - \frac{8\mu_{nw}}{r_l^2} \frac{dl_l}{dt} \right). \quad (20)$$

The Eqs. (19) and (20) can be written in dimensionless forms as,

$$1 = \frac{[x_l + M(1 - x_s)]}{1 + \eta^4} \left(\frac{dx_l}{dT} + \eta^2 \frac{dx_s}{dT} \right) + M(x_s - x_l) \left(\frac{dx_l}{dT} \right), \quad (21)$$

$$\beta - 1 = (x_s - x_l) \left(\frac{1}{\eta^2} \frac{dx_s}{dT} - M \frac{dx_l}{dT} \right), \quad (22)$$

where $x_i = \frac{l_i}{L}$ ($i = l, s$), $M = \frac{\mu_{nw}}{\mu_w}$, $\eta = \frac{r_s}{r_l}$, $\beta = \frac{P_{C1}}{P_{Cs}}$ and $T = \frac{P_{C1} r_l^2}{8\mu_w L^2} t$. By solving Eqs. (21) and (22), we obtain the distance traveled by the meniscus at different times during the imbibition process.

In Eq. (22), $\beta - 1$ is always positive as $\beta > 1$. Now, $\frac{1}{\eta^2} \left(\frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right)$ is the difference between the pressure gradients in the small and the large capillaries represented by lines AB and CD in Fig. 6(b). Till l_l the pressure in both capillaries is same. At l_l , the fluid transfers from the large capillary to the small capillary which should decrease the pressure gradient in the large capillary shown by CD in Fig. 6(b) and increases pressure gradient in the small capillary shown by the line AB in Fig. 6(b). Therefore, if $\frac{1}{\eta^2} \left(\frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right)$ is positive, then $x_s - x_l$ will always be positive. Thus, the one-dimensional model also predicts that x_s will be greater than x_l at all times. Further, we observe that by rearranging the equations in the one-dimensional model and adding Eqs. (21) and (22), we obtain,

$$2(1 + \beta\eta^4)T = (1 - M)(x_l^2 + \eta^2 x_s^2) + 2M(x_l + \eta^2 x_s). \quad (23)$$

We observe that Eq. (23) is analogous to Eq. (9), which describes the spontaneous imbibition of the interacting capillary system as a whole. We see that when the interacting capillary system is initially filled with a zero viscosity liquid for which the viscosity ratio $M = 0$, the sum of area of the capillary times the square of imbibition length in each of the capillaries is proportional to the time of imbibition, i.e., $(x_l^2 + \eta^2 x_s^2) \propto T$ which is different from the Washburn's law for a single capillary. For $M = 1$, we see that $(x_l + \eta^2 x_s) \propto T$ and for $M \gg 1$, the interacting capillary system takes infinite amount of time for the imbibition to start.

6. Comparison of VOF and one-dimensional model

Eqs. (21) and (22) are valid for a given viscous resistance and capillary pressure of the two interacting capillaries. For the VOF simulations in Section 4, the viscous resistance can be accurately calculated using the hydraulic radius. For accurate capillary pressure calculations for the interacting capillaries, we have used argument by Ouali et al. (2013). We consider the capillaries as open channels with the interaction AA' shown in Fig. 6(c) as open area for both the capillaries. At the meniscus, Ouali et al. (2013) relates the surface force F with the interfaces present in an open channel as,

$$\Delta F = \gamma(Y_{w,solid} \cos \delta_{w,solid} + Y_{w,nw} \cos \delta_{w,nw}), \quad (24)$$

where $Y_{w,solid}$ is the perimeter of the interface in contact with the solid wall, $Y_{w,nw}$ is the perimeter of the interface in contact with the non-wetting fluid, $\delta_{w,solid}$ is the contact angle made by the wetting fluid with the solid surface and $\delta_{w,nw}$ is the contact angle between the wetting and the non-wetting phases. For the small capillary, the contact angle between the wetting and the non-wetting phases at the second meniscus, $\delta_{w,nw}$, is considered π and $\delta_{w,solid}$ is 0. From the geometry of an interacting capillary system, we have $Y_{w,solid} = 2\pi r_i \left(1 - \frac{\theta_{r_i}}{\pi}\right)$, where $i = l, s$ and $2\theta_{r_i}$ is the arc angle as shown in Fig. 6(c). For the small capillary, $Y_{w,nw} \cos \delta_{w,nw} = -2r_s \sin \theta_{r_s}$ and Eq. (24) can be written as,

$$\Delta F_s = 2r_s \gamma \left(\pi \left(1 - \frac{\theta_{r_s}}{\pi}\right) - \sin \theta_{r_s} \right), \quad (25)$$

where F_s is the surface force in capillary of radius r_s . We know that for spontaneous imbibition, the force responsible for change in free energy is solely due to capillary pressure. Therefore, the capillary pressure for the small capillary, which is the ratio of the surface force to the cross section area, is given by,

$$Pc_{r_s} = \left(\frac{2\gamma}{r_s} \right) \frac{\pi \left(1 - \frac{\theta_{r_s}}{\pi}\right) - \sin \theta_{r_s}}{\pi \left(1 - \frac{\theta_{r_s}}{\pi}\right) + \frac{1}{2} \sin 2\theta_{r_s}}. \quad (26)$$

For the large capillary, the perimeter in contact with the non-wetting fluid $Y_{w,nw} = 0$ and with capillary surface $Y_{w,s} = 2\pi r_l \left(1 - \frac{\theta_{r_l}}{\pi}\right)$ for Eq. (24). The capillary force in the large capillary F_l is given by,

$$\Delta F_l = 2r_l \gamma \left(\pi \left(1 - \frac{\theta_{r_l}}{\pi}\right) \right), \quad (27)$$

The capillary pressure in the large capillary, Pc_{r_l} is given by,

$$Pc_{r_l} = \left(\frac{2\gamma}{r_l} \right) \frac{\pi \left(1 - \frac{\theta_{r_l}}{\pi}\right)}{\pi \left(1 - \frac{\theta_{r_l}}{\pi}\right) + \frac{1}{2} \sin 2\theta_{r_l}}. \quad (28)$$

In Table 1, we report the capillary pressure of the two capillaries given by VOF simulations and by using Eqs. (26) and (28) for the small and the large capillaries, respectively. Eqs. (26) and (28)

bring the capillary pressures closer to the observed capillary pressures from the VOF simulations in comparison to the ideal case capillary pressure calculated by the Young-Laplace equation.

In Fig. 7, we compare the imbibition lengths vs. time curve from the VOF simulations and one dimensional model using hydraulic radii and capillary pressures from Eqs. (26) and (28). The hydraulic radius for a capillary of radius r_i is calculated as the ratio of the cross-section area of the capillary to the perimeter of the capillary,

which is $\frac{r_i}{2} \left(1 + \frac{\sin 2\theta_{r_i}}{2\pi \left(1 - \frac{\theta_{r_i}}{\pi}\right)} \right)$. In addition to the case of equal viscosity

ratio of the resident to the imbibing fluid ($M = 1$), we have also run VOF simulations for $M = 0.67$ and 2, the comparison with one-dimensional model is shown in Appendix A. The comparison in Fig. 7 and Appendix A shows that our reduced order model is able to capture the imbibition phenomenon in interacting capillaries sufficiently well, especially for the cases when center-to-center distance between the capillaries is 1.495 mm and 1.44 mm. If, in our model, we use the capillary pressure from the VOF simulations, as shown in Fig. 3, then our model shows an excellent agreement with the simulations for the cases when center-to-center distance is 1.495 mm and 1.44 mm, as shown in Appendix B. However, when the interaction between the capillaries is more i.e., center-to-center distance is 1.25 mm, the capillaries behave close to a single capillary as can be seen from the insignificant menisci separation in the two capillaries in Figs. 3(d) and 7(d). When the center-to-center distance is 1.35 mm, the imbibition behaviour in the capillary system is between a single capillary and two interacting capillary system. An additional interaction parameter is required to exactly capture the imbibition behaviour using a reduced order model when the center-to-center distance is small, which requires further investigation. However, the present model is able to capture the trends well when the distinction between the two capillaries can still be made as shown by Fig. 7(c). In this work we focus on finding a model for an interacting capillary system when all the capillaries can be treated as separate capillaries that interact with each other.

To use the one-dimensional model in determining the properties of the porous medium, as earlier done by Purcell (1949), Dahle et al. (2005), Bartley and Ruth (1999, 2001) for bundle of capillaries, a model for multiple interacting capillaries is needed. Many arrangements of capillaries are possible, which may effect the imbibition dynamics unlike the models proposed by Dong et al. (2005, 2006) in which the capillaries are placed in the order of the radii. To this end, we now develop a one-dimensional model for three capillaries, which can be arranged in three ways to investigate the changes in flow behavior due to arrangement and the contrast in radii of the interacting capillaries.

7. Three interacting capillary system

Using the assumptions from the one-dimensional model for two capillaries and VOF simulations, we develop a lubrication approximation model for spontaneous imbibition in three interacting capillary system, for the three arrangements as shown in Figs. 8(a), 9(a) and 10(a). The following assumptions are used. (i) The pressure and pressure gradient is same in the capillaries where

Table 1
The capillary pressures in the small and the large capillaries as obtained from VOF simulations and the Eqs. (26) and (28) for the corresponding center-to-center distances between the capillaries.

Center-to-center distance (mm)	Pc_{r_l} (VOF)	Pc_{r_s} (VOF)	Pc_{r_l} from Eq. (28)	Pc_{r_s} from Eq. (26)
1.495	39.59	66.49	40.35	76.22
1.44	39.53	53.58	38.41	62.28
1.35	38.20	41.37	36.74	52.15
1.25	37.64	38.05	35.51	45.12

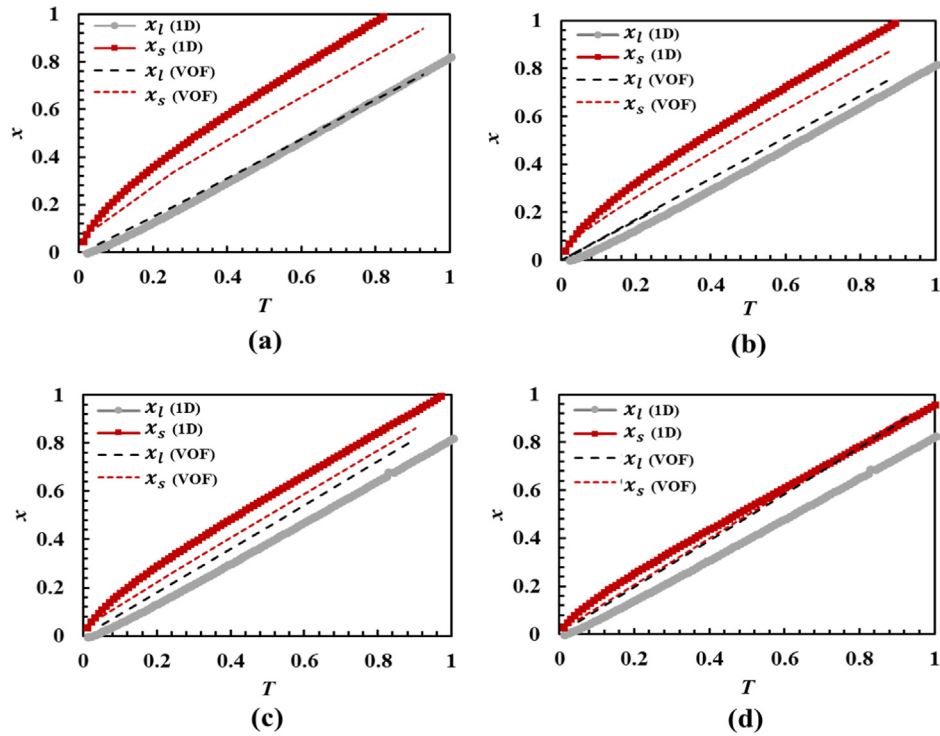


Fig. 7. Dimensionless distance vs dimensionless time comparison between the quasi one-dimensional model and VOF simulation for an interacting capillary system having capillaries of radii $r_l = 1$ mm, $r_s = 0.5$ mm and the center-to-center distance of (a) 1.495 mm, (b) 1.44 mm, (c) 1.35 mm and (d) 1.25 mm.

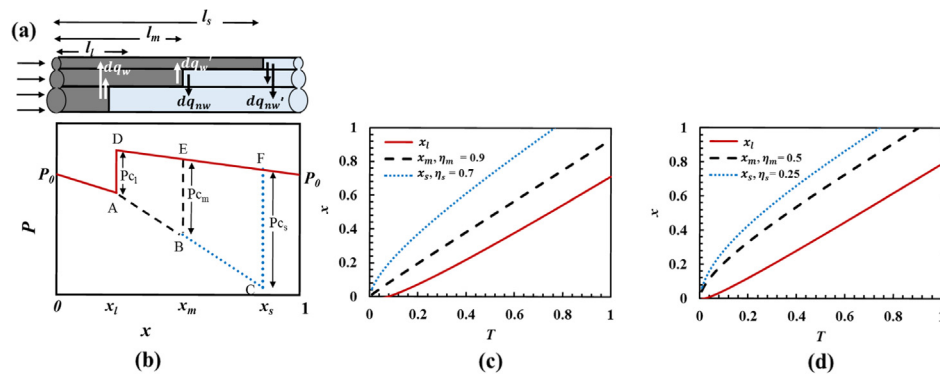


Fig. 8. An interacting three capillary system having the medium radius capillary (r_m) placed between capillaries of radii r_l and r_s , such that $r_s < r_m < r_l$. (a) Schematic of fluid transfer at menisci locations, (b) schematic of pressure vs. distance graph during imbibition, (c) dimensionless distance traveled by the meniscus vs. dimensionless time graph from the one-dimensional lubrication approximation model for $r_l = 1$ mm, $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$ and (d) dimensionless distance traveled by the meniscus vs. dimensionless time graph for $r_l = 1$ mm, $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$ from the one-dimensional lubrication approximation model. The meniscus position in the small radius capillary is always ahead of the medium radius capillary, which is ahead of the large radius capillary.

two or more capillaries have the same fluid which is continuous. In Figs. 8(b), 9(b) and 10(b), we show the anticipated pressures profiles during imbibition for each of the arrangements shown in Figs. 8(a), 9(a) and 10(a), respectively. (ii) The fluid cross-flow between the capillaries occurs only at the menisci. (iii) The fluid cross-flow of wetting fluid occurs from a larger to a smaller capillary and cross flow of the non-wetting fluid occurs from a smaller to a larger capillary. The schematics in Figs. 8(a), 9(a), 10(a) shows the fluid cross-flow directions near the menisci. In developing the one-dimensional model, we consider capillaries of length L and hydraulic radii r_s , r_m and r_l , such that $r_s < r_m < r_l$ referred to as small, medium and large capillaries, respectively. In the arrangement shown in Fig. 8(a), case I, capillary of radius r_m is in the middle. In the arrangement shown in Fig. 9(a), case II, capillary of radius r_s is in the middle. In the arrangement shown in Fig. 10

(a), case III, capillary of radius r_l is in the middle. The capillary system is initially filled with a non-wetting fluid of viscosity μ_{nw} at time $t = 0$. One end of the capillary system is brought in contact with a wetting fluid of viscosity μ_w . After time t , we assume that the wetting fluid has imbibed till a length l_l in the large capillary, l_m in the medium capillary and l_s in the small capillary.

To develop a one-dimensional lubrication approximation model, we use Hagen-Poiseuille's law for pressure loss as used in Section 5.

7.1. Case I

In the case shown in Fig. 8(a), we see that l_s is greater than l_m , and l_m is greater than l_l . This is because, the pressure should be

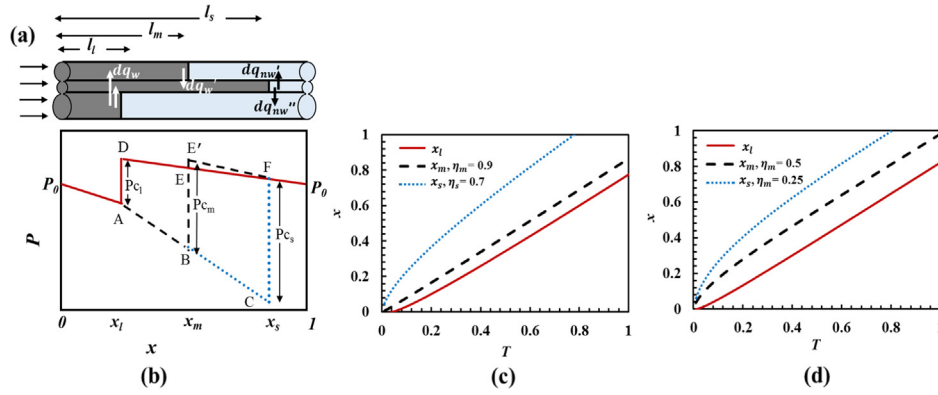


Fig. 9. An interacting three capillary system having the small radius capillary (r_s) placed between capillaries of radii r_l and r_m , such that $r_s < r_m < r_l$. (a) Schematic of fluid transfer at menisci locations, (b) schematic of pressure vs distance graph during imbibition, (c) dimensionless distance traveled by the meniscus vs. dimensionless time graph from the one-dimensional lubrication approximation model for $r_l = 1$ mm, $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$ and (d) dimensionless distance traveled by the meniscus vs. dimensionless time graph for $r_l = 1$ mm, $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$ from the one-dimensional lubrication approximation model. The meniscus position in the small radius capillary is always ahead of the medium radius capillary, which is ahead of the large radius capillary.

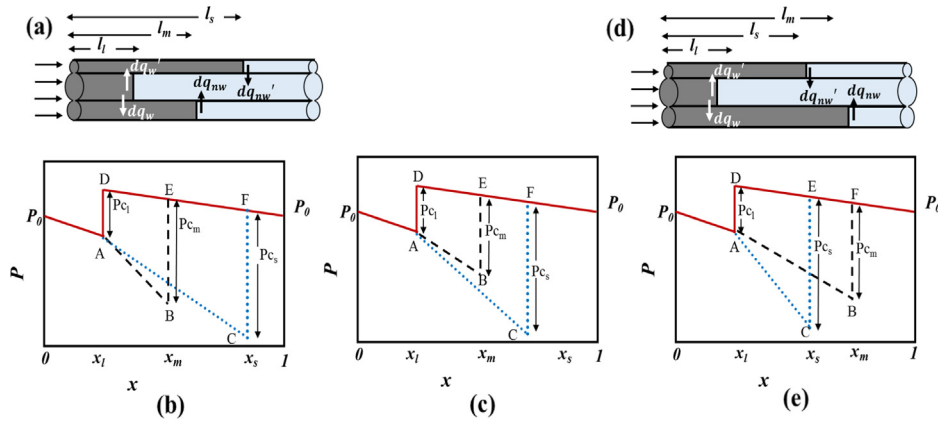


Fig. 10. An interacting three capillary system having the large radius capillary (r_l) placed between capillaries of radii r_m and r_s , such that $r_s < r_m < r_l$. (a) Schematic of fluid transfer at menisci locations for $l_s > l_m > l_l$, (b) schematic of pressure vs distance graph during imbibition for $l_s > l_m > l_l$ where, the pressure gradient in the medium capillary is greater than the pressure gradient in the small capillary after x_l , (c) schematic of pressure vs distance graph during imbibition for $l_s > l_m > l_l$ where, the pressure gradient in the small capillary is greater than the pressure gradient in the medium capillary after x_l , (d) schematic of fluid transfer at menisci locations for $l_m > l_s > l_l$ and (e) schematic of pressure vs distance graph during imbibition for $l_m > l_s > l_l$ where, the pressure gradient in the small capillary is greater than the pressure gradient in the medium capillary after x_l .

same in all the capillaries till the first meniscus. At the first meniscus, the wetting fluid should transfer from large capillary to the smaller capillaries due to capillary suction. This increases the pressure gradient in the two smaller capillaries and decreases the pressure gradient in the largest capillary. If the viscosity ratio is favorable, then the slope of line AB should be greater than slope of line DE, as shown in Fig. 8(b). If so, the wetting fluid should travel more in the two smaller capillaries in comparison to the largest capillary. This would make capillary pressure jump BE more than the capillary pressure jump AD, i.e., $P_{cm} > P_{cl}$, which is true for the capillary radii assumed. By the same principle, the wetting fluid should travel fastest in the smallest capillary.

Using the above argument and Hagen-Poiseuille's law, the one-dimensional model for the menisci lengths is given by,

$$P_{cl} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (29)$$

$$P_{cm} - P_{cl} = (l_m - l_l) \left(\frac{8\mu_w}{r_s^4 + r_m^4} \left(r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (30)$$

$$P_{cs} - P_{cm} = (l_s - l_m) \left(\frac{8\mu_w}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right) \right), \quad (31)$$

where P_{ci} for $i = l, m, s$ are the true capillary pressures experienced by the wetting fluid in the capillaries and r_l, r_m, r_s are the hydraulic radii of the capillaries. The terms $\frac{dl_i}{dt}$ is the meniscus velocity in a capillary of hydraulic radius r_i . The details of the derivation are given in Appendix C. In dimensionless form these equations are,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_m - x_l) \left(\frac{dx_l}{dT} \right) + \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right), \quad (32)$$

$$\beta_m - 1 = (x_m - x_l) \left(\frac{1}{\eta_m^4 + \eta_s^4} \left(\eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right), \quad (33)$$

$$\beta_s - \beta_m = (x_s - x_m) \left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - \frac{M}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right) \right), \quad (34)$$

where the dimensionless lengths are $x_i = \frac{l_i}{L}$ for $i = l, m, s$; the viscosity ratio is $M = \frac{\mu_w}{\mu_{nw}}$, $\beta_i = \frac{P_{ci}}{P_{cl}}$, $\eta_i = \frac{r_l}{r_i}$ and $T = t \frac{P_{cl} r_l^2}{8\mu_w L^2}$.

We observe from Eq. (33) that the left hand side $(\beta_m - 1) > 0$. On the right hand side, $\frac{1}{\eta_m^4 + \eta_s^4} (\eta_m^2 \frac{dx_m}{dt} + \eta_s^2 \frac{dx_s}{dt}) - M \left(\frac{dx_l}{dt} \right) > 0$, which is the difference between the slopes AB and DE of Fig. 8(b). This implies that x_m will always be greater than x_l . Similarly, in Eq. (34), we see that $(\beta_s - \beta_m)$ is positive and $\left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dt} \right) - \frac{M}{1 + \eta_m^4} \left(\frac{dx_l}{dt} + \eta_m^2 \frac{dx_m}{dt} \right) \right)$ is the difference between the slopes of lines BC and EF, which should be positive as shown in Fig. 8(b). Therefore, x_s is always greater than x_m .

In Fig. 8(c), we show the menisci lengths with time for a capillary system, where $r_l = 1$ mm, $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$ considering Young-Laplace equation for capillary pressures. Here, $(\eta_m + \eta_s)$ is greater than unity. In Fig. 8(d), we show the menisci lengths with time for a capillary system, where $r_l = 1$ mm, $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$ considering Young-Laplace equation for capillary pressures. Here, $(\eta_m + \eta_s)$ is less than unity. In both the cases, we see that $x_s > x_m > x_l$.

7.2. Case II

In the case shown in Fig. 9(a), we show that the meniscus in the smallest capillary leads, followed by the meniscus in the medium capillary, followed by the meniscus in the largest capillary. As we have established, the pressure gradient till the first meniscus should be same in all the capillaries, after which the fluid transfers from the largest capillary to the small and the medium capillaries due to capillary suction. As explained for case I, the slope of AB should be greater than the slope of DE and thus $P_{Cm} > P_{Cl}$ is also satisfied, shown in Fig. 9(b). This implies that l_l is less than l_m and l_s , if the viscosity ratio is favorable. Now, till the second meniscus, the pressure is same in the medium and the small capillaries. The fluid transfer occurs at the second meniscus from the medium to small capillary due to capillary suction. This implies, slope of BC is more than the slope in the medium and the large capillaries, i.e., EF and E'F and thus $P_{Cs} > P_{Cm}$ is also satisfied as shown in Fig. 9(b). Therefore, the second meniscus should be in the medium capillary only. However, EF and E'F may not be equal. Using this, the one-dimensional model for case II is (details in Appendix C),

$$P_{Cl} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right), \quad (35)$$

$$P_{Cs} - P_{Cl} = \frac{8\mu_w(l_m - l_l)}{r_s^4 + r_m^4} \left(r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_w(l_s - l_m)}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) \quad (36)$$

$$P_{Cs} - P_{Cm} = (l_s - l_m) \left(\frac{8\mu_w}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_m^2} \left(\frac{dl_m}{dt} \right) \right), \quad (37)$$

where P_{Ci} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the meniscus velocity respectively in a capillary of hydraulic radius r_i . In dimensionless form the Eqs. (35)–(37) are,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dt} + \eta_m^2 \frac{dx_m}{dt} + \eta_s^2 \frac{dx_s}{dt} \right) + M(x_s - x_l) \left(\frac{dx_l}{dt} \right), \quad (38)$$

$$\beta_s - 1 = \frac{(x_m - x_l)}{\eta_m^4 + \eta_s^4} \left(\eta_m^2 \frac{dx_m}{dt} + \eta_s^2 \frac{dx_s}{dt} \right) + \frac{(x_s - x_m)}{\eta_s^2} \left(\frac{dx_s}{dt} \right) - M(x_s - x_l) \left(\frac{dx_l}{dt} \right), \quad (39)$$

$$\beta_s - \beta_m = (x_s - x_m) \left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dt} \right) - \frac{M}{\eta_m^2} \left(\frac{dx_m}{dt} \right) \right), \quad (40)$$

where the dimensionless parameters are $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{Ci}}{P_{Cl}}$, $\eta_i = \frac{r_i}{r_l}$ and $T = t \frac{P_{Cl} r_l^2}{8\mu_w L^2}$.

We see from, Eq. (40), that the left hand side is positive. On the right hand side, $\left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dt} \right) - \frac{M}{\eta_m^2} \left(\frac{dx_m}{dt} \right) \right)$ is the difference between the slopes of lines BC and E'F, which is positive as observed from Fig. 9(b). Therefore, x_s is always greater than x_m . We see that x_l is always the least as observed by the pressure graph in Fig. 8(b). Therefore, in this case also $x_s > x_m > x_l$ at all times. We use the same parameters of the capillaries as in Section 7.1 and show the menisci positions of the capillaries. In Figs. 9(c), we show the menisci lengths with time for a capillary system, where $r_l = 1$ mm, $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$ considering Young-Laplace equation for capillary pressures. In Fig. 8(d), we show the menisci lengths with time for a capillary system, where $r_l = 1$ mm, $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$ considering Young-Laplace equation for capillary pressures. In both cases, we see that $x_s > x_m > x_l$.

7.3. Case III

In Fig. 10(a), we show the anticipated menisci lengths for the case where the largest capillary is in between the small and medium capillaries. Till the first meniscus, the pressure in the capillaries is same as shown in Fig. 10(b, c, e). At the first meniscus, the wetting fluid transfers from the large capillary to the medium and the small capillaries, implying that the meniscus in the large capillary will lag behind the other two menisci. The pressure gradient DE will now be less than the pressure gradient AB and AC, and thus $P_{Cm} > P_{Cl}$ and $P_{Cs} > P_{Cl}$ are also satisfied as shown in Figs. 10(b, c, e). However, if the slope of AB is more or less than AC, is not known, as the wetting fluid in the medium and the small capillaries do not interact after l_l . The capillary suction in the small capillary is more, which will increase the velocity and the pressure gradient in the small capillary after l_l as shown in Fig. 10(c). Therefore, slope of AB should be less than AC as shown in Fig. 10(c). However, this does not affect the governing equations and the menisci lengths are given by,

$$P_{Cl} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (41)$$

$$P_{Cm} - P_{Cl} = (l_m - l_l) \left(\frac{8\mu_w}{r_m^2} \left(\frac{dl_m}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (42)$$

$$P_{Cs} - P_{Cl} = \frac{8\mu_w(l_s - l_l)}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) - \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (43)$$

where P_{Ci} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the menisci velocity respectively in a capillary of hydraulic radius r_i . The details of the derivation are given in Appendix C. The Eqs. (41)–(43) are written in dimensionless forms as,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dt} + \eta_m^2 \frac{dx_m}{dt} + \eta_s^2 \frac{dx_s}{dt} \right) + M(x_m - x_l) \left(\frac{dx_l}{dt} \right) + \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dt} + \eta_m^2 \frac{dx_m}{dt} \right), \quad (44)$$

$$\beta_m - 1 = (x_m - x_l) \left(\frac{1}{\eta_m^2} \left(\frac{dx_m}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right). \quad (45)$$

$$\beta_s - 1 = \frac{(x_s - x_l)}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - M(x_m - x_l) \left(\frac{dx_l}{dT} \right) - \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right), \quad (46)$$

where $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_l}{r_i}$ and $T = t \frac{P_{C_l} r_l^2}{8 \mu_w L^2}$.

From Eq. (45), we see that the left hand side of the equation is positive and on the right hand side, $\frac{1}{\eta_m^2} \left(\frac{dx_m}{dT} \right) - M \left(\frac{dx_l}{dT} \right) > 0$ as the slope of the line AB is greater than the slope of the line DE in Fig. 10(c). This tells us that x_m will always be greater than x_l . However, Eq. (46) does not preclude the case where x_s is less than x_l . Fig. 10(d) and (e) describe a situation where x_m is greater than x_s and the condition of pressure gradient used above are all satisfied. Using this condition, the governing equations of the menisci are,

$$P_{C_l} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_m)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_m - l_s)}{r_l^2 + r_s^2} \left(r_l^2 \frac{dl_l}{dt} + r_s^2 \frac{dl_s}{dt} \right), \quad (47)$$

$$P_{C_m} - P_{C_l} = \frac{8\mu_w(l_m - l_l)}{r_m^2} \left(\frac{dl_m}{dt} \right) - \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) - \frac{8\mu_{nw}(l_m - l_s)}{r_l^2 + r_s^2} \left(r_l^2 \frac{dl_l}{dt} + r_s^2 \frac{dl_s}{dt} \right). \quad (48)$$

$$P_{C_s} - P_{C_l} = (l_s - l_l) \left(\frac{8\mu_w}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (49)$$

where P_{C_i} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the meniscus velocity respectively in a capillary of hydraulic radius r_i . A detailed derivation is given in Appendix C. The Eqs. (47)–(49) are written in dimensionless forms as,

$$1 = \frac{x_l + M(1 - x_m)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_s - x_l) \left(\frac{dx_l}{dT} \right) + \frac{M(x_m - x_s)}{1 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_s^2 \frac{dx_s}{dT} \right), \quad (50)$$

$$\beta_m - 1 = \frac{(x_m - x_l)}{\eta_m^2} \left(\frac{dx_m}{dT} \right) - M(x_s - x_l) \left(\frac{dx_l}{dT} \right) - \frac{M(x_m - x_s)}{1 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_s^2 \frac{dx_s}{dT} \right), \quad (51)$$

$$\beta_s - 1 = (x_s - x_l) \left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right), \quad (52)$$

where $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_l}{r_i}$ and $T = t \frac{P_{C_l} r_l^2}{8 \mu_w L^2}$.

We use two radii systems to show the capillary imbibition. In Fig. 11(a), we show a system with $r_l = 1$ mm, $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$ considering Young-Laplace equation for capillary pressures. For this case, the meniscus in the small capillary leads and we use Eqs. (44)–(46) to find the menisci lengths. If we had used Eqs. (50)–(52), we still get the meniscus in the small capillary leading. However, the quantitative results will not be accurate due to the pressure gradients used to derive the Eqs. (50)–(52) are for $x_m > x_s > x_l$. For a system with $r_l = 1$ mm, $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$ considering Young-Laplace equation for capillary pressures, the meniscus in the medium capillary leads and we use Eqs. (50)–(52) to determine the menisci lengths. If the Eqs. (44)–(46) had been used, we get a similar qualitative prediction. However, the quantitatively, the results are inaccurate due to the pressure gradients used in developing Eqs. (44)–(46). Therefore, we see that when the largest capillary is placed between the small and the medium capillaries, and if $(\eta_m + \eta_s) > 1$, then $x_s > x_m > x_l$. However, if $(\eta_m + \eta_s) < 1$, then $x_m > x_s > x_l$. The case $(\eta_m + \eta_s) < 1$ represents a more heterogeneous system where we see anomalous result that the meniscus in the medium capillary leads.

This explains that in an interacting capillary system consisting of multiple capillaries placed in random order, it is possible that the meniscus in the smallest radius capillary may lag behind the menisci of capillaries with larger radii.

We observe that by suitable rearrangement of the one-dimensional model equations for each of the cases I, II, III and adding them, we obtain,

$$2(1 + \beta_m \eta_m^4 + \beta_s \eta_s^4)T = (1 - M)(x_l^2 + \eta_m^2 x_m^2 + \eta_s^2 x_s^2) + 2M(x_l + \eta_m^2 x_m + \eta_s^2 x_s). \quad (53)$$

Eq. (53) is analogous to Eq. (9). Eq. (53) explains that irrespective of the arrangement of capillaries and the menisci locations, the spontaneous imbibition of the interacting capillary system as a whole follows same behavior and varies from Washburn like diffusive behavior. However, this describes that sum of volume imbibed in individual capillary times the length is proportional to the time, if $M = 0$. Eq. (53) also explains that for $M \gg 1$, the time needed for imbibition to start is very long, which explains that the model is applicable for favorable viscosity ratio.

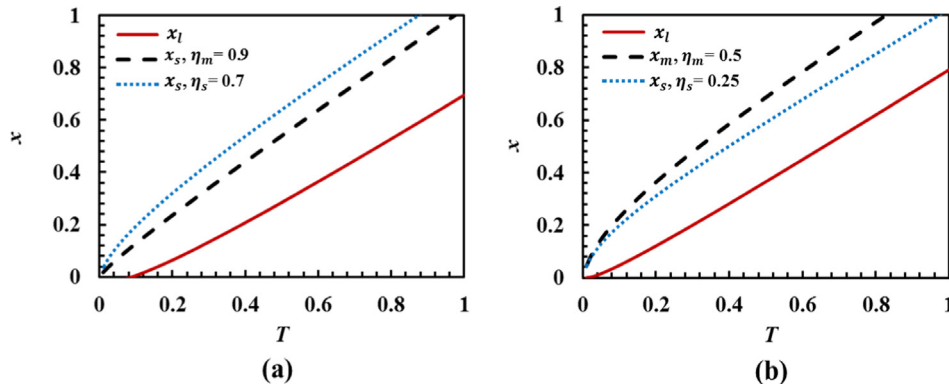


Fig. 11. An interacting three capillary system having the large radius capillary (r_l) placed between capillaries of radii r_m and r_s , such that $r_s < r_m < r_l$. Dimensionless distance vs. dimensionless time graph determined from the one-dimensional lubrication approximation model for (a) $x_s > x_m > x_l$ for $M = 1$, $\eta_m = 0.9$, $\eta_s = 0.7$, $\beta_m = 1.11$, $\beta_s = 1.42$, (b) $x_m > x_s > x_l$ for $M = 1$, $\eta_m = 0.5$, $\eta_s = 0.25$, $\beta_m = 2$, $\beta_s = 4$.

8. Conclusions

In conclusion, we have used VOF simulations to find the approximations for a quasi one-dimensional model for interacting capillaries. We can use pressure equilibration in the capillaries in the direction perpendicular to the axial interaction where same fluid is present and is continuous. Using the streamlines from VOF simulation, we described that the wetting fluid transfers from the large capillary to the small capillary and the non-wetting fluid transfers from the small capillary to the large capillary near the menisci, which can be considered at the menisci locations for one-dimensional model. Using these observations, we developed a quasi one-dimensional lubrication approximation model for two interacting capillaries which is an excellent approximation of the spontaneous imbibition in two interacting capillaries shown by VOF simulations. For the interacting three capillary system, three arrangements are possible. We developed a lubrication approximation model for all the three arrangements and show that the spontaneous imbibition is dependent on the arrangement as well as the contrast in the radii of the three capillaries. More precisely, when the large capillary is in between the small and the medium capillaries, the pressure in the wetting phase does not equilibrate. This leads to a variation in the menisci positions, which is dependent on the radii of capillaries.

The one-dimensional model can be generalized for a multiple interacting capillary system and can be used to determine the properties of an ideal porous medium to understand the dynamics of spontaneous imbibition in interacting pores of a porous matrix. The results from this study can be used in micro reactors, microfluidic devices and in lab-on-chip devices where the effect of interaction between various capillaries can be included for better designs. The imbibition dynamics in interacting pores will also help in understanding the spontaneous imbibition at pore scale in fractured oil reservoirs.

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Appendix A. Comparison of VOF simulations with the numerical solution of the two interacting capillary system for unequal viscosity of resident and imbibing fluid

Fig. A.12 compares the imbibed dimensionless length vs dimensionless time using VOF simulations of two interacting capillaries with center-to-center distance of 1.495 mm and our one-dimensional model for viscosity ratio $M = 0.67$ and 2.

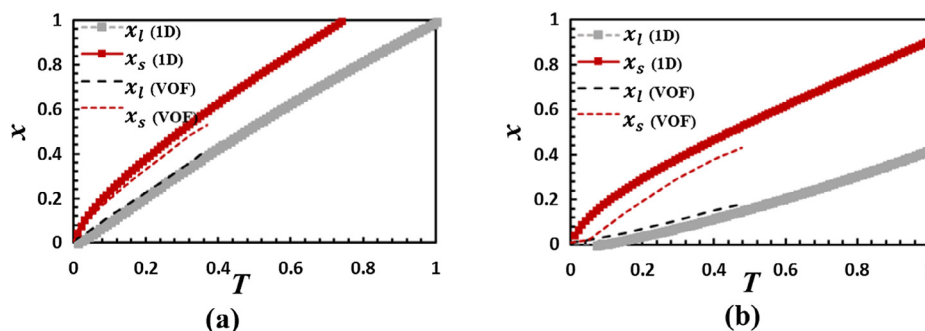


Fig. A.12. Dimensionless length vs time comparison of the VOF simulations and one-dimensional model for two interacting capillaries of radii 1 mm and 0.5 mm, with center-to-center distance of 1.495 mm and viscosity ratio (a) $M = 0.667$, (b) $M = 2$.

Appendix B. Numerical solution to the two interacting capillary system using the capillary pressures from VOF simulations

Fig. B.13 shows the comparison of the dimensionless meniscus displacement in the two interacting capillaries with capillary pressures obtained using VOF simulations used in our one-dimensional model. For the one-dimensional model, we have used capillary pressures from Fig. 3. The one dimensional model shows an excellent agreement for the cases when the center-to-center distance between the two capillaries is 1.495 mm and 1.44 mm. When the center to center distance is 1.35 mm, our model is able to predict the trend sufficiently well, however, the capillaries now do not behave entirely separately. Therefore, an interaction parameter is needed in the one-dimensional model for this case, which requires further investigation. For the case when center to center distance is 1.25 mm, the interacting capillary system behaves close to a single capillary. In this case also, our one-dimensional model is able to infer the closeness of the menisci, however, deviates from the exact match due to the change in shape of the menisci in both the capillaries as shown in Fig. 4(d).

Appendix C. Equation development for three interacting capillary system

C.1. Case I

We show in the schematic of Fig. 8(a) that for this arrangement, the medium capillary is placed in between the small and the large capillaries. We infer from the VOF simulations that the pressure in all three capillaries is same till the first meniscus, i.e., till l_i . Then from l_i to l_m , the pressure in the medium and the small capillaries is same as they are filled with wetting fluid, which is represented by the line AB, shown in Fig. 8(b). Therefore, the only possibility of the menisci locations is $l_i < l_m < l_s$ as shown in Fig. 8(a). From Hagen-Poiseuille law, the viscous loss in the large capillary till l_i is,

$$P_w(l_i, t) - P_o = -\frac{8\mu_w l_i}{r_l^2} U_l(t). \quad (C.1)$$

In Eq. (C.1), μ_w is the viscosity of the wetting phase, $U_l(t)$ is the wetting phase velocity in the large capillary, P_o is the atmospheric pressure at the inlet and $P_w(l_i, t)$ is the pressure in the wetting phase at the meniscus in the large capillary. As the pressure drop from the inlet till l_i is same in all the three capillaries, it implies that $\frac{U_l(t)}{r_l^2} = \frac{U_m(t)}{r_m^2} = \frac{U_s(t)}{r_s^2}$, where, $U_m(t)$ and $U_s(t)$ are the velocities of the wetting phase in the medium and the small capillaries before l_i . The wetting fluid transfers from the large capillary to the medium capillary and the small capillary at the first meniscus, which increases the flow rate in the medium and the small capillaries after

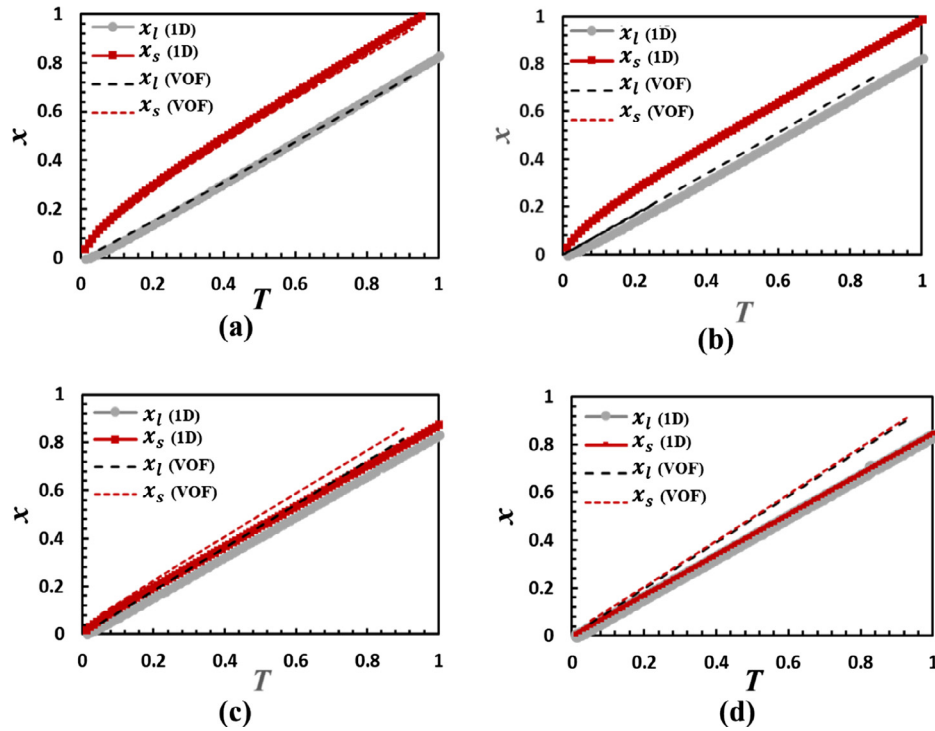


Fig. B.13. Dimensionless distance vs dimensionless time comparison between the VOF simulations and numerical solution to the quasi one-dimensional model solved using the capillary pressures obtained from VOF simulations for an interacting capillary system having capillaries of radii $r_l = 1$ mm, $r_s = 0.5$ mm and the center-to-center distance of (a) 1.495 mm, (b) 1.44 mm, (c) 1.35 mm and (d) 1.25 mm.

l_l . We assume that the volumetric flow rate transfer from the large capillary is dq_w as shown in Fig. 8(a). As the fluid transfer occurs before the meniscus at l_l , the meniscus velocity in the large capillary is given by

$$\frac{dl_l}{dt} = U_l(t) - \frac{dq_w}{\pi r_l^2}. \quad (\text{C.2})$$

The pressure drop after l_l in the large capillary is due to the resident non-wetting fluid which is given as,

$$P_{nw}(l_m, t) - P_{nw}(l_l, t) = -\frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} \right), \quad (\text{C.3})$$

where $P_{nw}(l_i, t)$ is the non-wetting phase pressure at position l_i ($i = l, m, s$). Let the fraction of fluid transfer in the medium capillary be f . Therefore, the fluid transfer in the medium capillary is fdq_w and that in the small capillary is $(1-f)dq_w$. We know that, the pressure drop from l_l to l_m is same in the capillaries of radii r_m

and r_s , i.e., $\frac{U_m(t) + \frac{fdq_w}{\pi r_m^2}}{r_m^2} = \frac{U_s(t) + \frac{(1-f)dq_w}{\pi r_s^2}}{r_s^2}$, which gives us that $f = \frac{r_m^2}{r_m^2 + r_s^2}$. From l_l to l_m , the medium and the small capillaries are filled with wetting fluid and have a pressure drop shown by line AB in Fig. 8(b), which is given by,

$$P_w(l_m, t) - P_w(l_l, t) = -\frac{8\mu_w(l_m - l_l)}{r_m^2} \left(U_l(t) \frac{r_m^2}{r_l^2} + \frac{fdq_w}{\pi r_m^2} \right). \quad (\text{C.4})$$

In Eq. (C.4), $P_w(l_i, t)$ is the wetting phase pressure at position l_i . At l_m , some of the wetting fluid in the medium capillary transfers into the small capillary as shown in Fig. 8(a), which is dq'_w . As the wetting fluid transfer is occurs just before the meniscus, the velocity of the meniscus in the medium capillary is given as,

$$\frac{dl_m}{dt} = U_m(t) + \frac{fdq_w}{\pi r_m^2} - \frac{dq'_w}{\pi r_m^2}. \quad (\text{C.5})$$

After the meniscus in the medium capillary, there is non-wetting fluid transfer of dq_{nw} from the medium capillary to the large capillary. The fluid transfer occurs such that, the pressure drop in the large and the medium capillaries are the same from l_m to l_s as shown in Fig. 8(b), i.e., $\frac{U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq_{nw}}{\pi r_l^2}}{r_l^2} = \frac{U_m(t) + \frac{fdq_w}{\pi r_m^2} - \frac{dq_{nw}}{\pi r_m^2}}{r_m^2}$. The pressure drop of non-wetting fluid from l_m till l_s is given by,

$$P_{nw}(l_s, t) - P_{nw}(l_m, t) = -\frac{8\mu_{nw}(l_s - l_m)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} + \frac{dq_{nw}}{\pi r_l^2} \right). \quad (\text{C.6})$$

After the wetting phase transfer dq'_w from the medium to the small capillary, the velocity in the small capillary is given as,

$$\frac{dl_s}{dt} = U_s(t) + \frac{(1-f)dq_w}{\pi r_s^2} + \frac{dq'_w}{\pi r_s^2}. \quad (\text{C.7})$$

The pressure drop in the small capillary from l_m to l_s is given as,

$$P_w(l_s, t) - P_w(l_m, t) = -\frac{8\mu_w(l_s - l_m)}{r_s^2} \left(U_l(t) \frac{r_s^2}{r_l^2} + \frac{(1-f)dq_w}{\pi r_s^2} + \frac{dq'_w}{\pi r_s^2} \right). \quad (\text{C.8})$$

Immediately after the meniscus in the small capillary, the non-wetting fluid transfer occurs into the large and the medium capillaries such that the pressure drop across all the three capillaries is same from l_s till the outlet as shown in Fig. 8(b), which is given as,

$$P_o - P_{nw}(l_s, t) = -\frac{8\mu_{nw}(L - l_s)}{r_l^2} U_l(t). \quad (\text{C.9})$$

We know that the Young-Laplace pressure jump in a capillary of radius r_i is P_{Ci} , which is given as,

$$P_{nw}(l_i, t) - P_w(l_i, t) = P_{Ci}, \quad (\text{C.10})$$

where $i = l, m, s$ for capillaries of radii r_l, r_m, r_s respectively. By substituting the values of menisci velocities and eliminating fluid transfers dq_w, dq'_w, dq_{nw} , and by introducing the capillary pressures calculated from Eq. (C.10), we obtain the equations governing the flow,

$$P_{C_l} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (C.11)$$

$$P_{C_m} - P_{C_l} = (l_m - l_l) \left(\frac{8\mu_w}{r_s^4 + r_m^4} \left(r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (C.12)$$

$$P_{C_s} - P_{C_m} = (l_s - l_m) \left(\frac{8\mu_w}{r_s^4} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right) \right), \quad (C.13)$$

where P_{C_i} is the true capillary pressure in the capillary of hydraulic radius r_i , for $i = l, m, s$. The term $\frac{dl_i}{dt}$ is the meniscus velocity in a capillary of hydraulic radius r_i . In dimensionless form, these equations are,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_m - x_l) \left(\frac{dx_l}{dT} \right) + \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right), \quad (C.14)$$

$$\beta_m - 1 = (x_m - x_l) \left(\frac{1}{\eta_m^4 + \eta_s^4} \left(\eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right), \quad (C.15)$$

$$\beta_s - \beta_m = (x_s - x_m) \left(\frac{1}{\eta_s^4} \left(\frac{dx_s}{dT} \right) - \frac{M}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right) \right), \quad (C.16)$$

where the dimensionless lengths are $x_i = \frac{l_i}{L}$ for $i = l, m, s$; the viscosity ratio is $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_i}{r_l}$ and $T = t \frac{P_{C_l} \mu_l^2}{8\mu_w L^2}$.

C.2. Case II

In this arrangement, we see from the schematic of Fig. 9(a) that the small capillary is placed between the large and the medium capillaries. From the inlet till the meniscus in the large capillary, we show from Fig. 9(b) that the pressure drop in all the three capillaries is the same. The pressure drop till the first meniscus is given by Eq. (C.1) and the velocities of wetting fluid in the capillaries is related as $\frac{U_l(t)}{r_l^2} = \frac{U_m(t)}{r_m^2} = \frac{U_s(t)}{r_s^2}$. In Fig. 9(a), we show the schematic of menisci positions and fluid transfers and a wetting fluid transfer of dq_w occurs from the large capillary to the medium and the small capillaries due to capillary suction after which the velocity of the meniscus in the large capillary is explained by Eq. (C.2). The transferred wetting phase dq_w splits up into the medium and the small capillaries in fractions of $f dq_w$ and $(1 - f) dq_w$ respectively, thus increasing the pressure gradient in these capillaries after l_l , shown in Fig. 9(b) as AB. The fraction $f = \frac{r_m^2}{r_m^2 + r_s^2}$, such that the pressure drop across the medium and the small capillaries is

the same from l_l to l_m , i.e., $\frac{U_m(t) + \frac{f dq_w}{r_m^2}}{r_m^2} = \frac{U_s(t) + \frac{(1-f) dq_w}{r_s^2}}{r_s^2}$ is valid, and the pressure drop in the capillaries after the first meniscus till l_m is

given by Eq. (C.4). After the fluid transfer dq_w , the pressure drop in the non-wetting phase till l_s is given by,

$$P_{nw}(l_s, t) - P_{nw}(l_l, t) = - \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} \right), \quad (C.17)$$

as the flow rate in the large capillary remains constant from l_l till l_s . As the capillaries of radii r_m and r_s are hydrodynamically interacting, the wetting fluid transfers from the medium capillary to the small capillary at l_m due to capillary suction as shown in the schematic of Fig. 9(a), represented by dq'_w . Thus the flow rate in the small capillary increases, causing an increase in the pressure drop of wetting fluid after l_m . The velocity of the meniscus in the medium capillary is determined by the fluid transfer dq'_w and is given by Eq. (C.5), while the meniscus velocity in the small capillary is explained by Eq. (C.7). The pressure drop of wetting phase after l_m is given by Eq. (C.8) and it should be noted that the pressure drop of the non-wetting phase from l_m to l_s in the medium capillary is different from the large capillary as the non-wetting fluid in these capillaries is discontinuous. The pressure drop of non-wetting fluid in the medium capillary is given as,

$$P'_{nw}(l_s, t) - P'_{nw}(l_m, t) = - \frac{8\mu_{nw}(l_s - l_m)}{r_m^2} \left(U_l(t) \frac{r_m^2}{r_l^2} + f \frac{dq_w}{\pi r_m^2} - \frac{dq'_w}{\pi r_m^2} \right), \quad (C.18)$$

where $P'_{nw}(l_i, t)$ is the pressure of the non-wetting fluid in the medium capillary at l_i ($i = m, s$). After l_s , the non-wetting fluid transfers from the small capillary to the large and the medium capillaries such that the pressure drop in all the three capillaries is same till the exit, which is given by Eq. (C.9). The capillary pressure in each capillary is the difference between the non-wetting and wetting phase pressures at the menisci as given by Eq. (C.10). We use the pressure drops in each of the capillaries at different menisci locations to substitute for the capillary pressures. The fluid transfers and the velocities $U_l(t), U_m(t), U_s(t)$ are substituted for the menisci velocities and we obtain the governing equations,

$$P_{C_l} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right), \quad (C.19)$$

$$P_{C_s} - P_{C_l} = \frac{8\mu_w(l_m - l_l)}{r_s^4 + r_m^4} \left(r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_w(l_s - l_m)}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) \quad (C.20)$$

$$P_{C_s} - P_{C_m} = (l_s - l_m) \left(\frac{8\mu_w}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_m^2} \left(\frac{dl_m}{dt} \right) \right), \quad (C.21)$$

where P_{C_i} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the meniscus velocity respectively in a capillary of hydraulic radius r_i . In dimensionless form the Eqs. (35)–(37) are,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_s - x_l) \left(\frac{dx_l}{dT} \right), \quad (C.22)$$

$$\beta_s - 1 = \frac{(x_m - x_l)}{\eta_m^4 + \eta_s^4} \left(\eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + \frac{(x_s - x_m)}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - M(x_s - x_l) \left(\frac{dx_l}{dT} \right), \quad (C.23)$$

$$\beta_s - \beta_m = (x_s - x_m) \left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - \frac{M}{\eta_m^2} \left(\frac{dx_m}{dT} \right) \right), \quad (C.24)$$

where the dimensionless parameters are $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_i}{r_l}$ and $T = t \frac{P_{C_l} r_l^2}{8\mu_w L^2}$.

C.3. Case III

Let us consider the case when the meniscus in the small capillary leads, i.e., $l_s > l_m > l_l$ as shown in Fig. 10(a)–(c). The pressure gradient in all the three capillaries is the same from the inlet till the first meniscus, which is given by Eq. (C.1) and therefore, $\frac{U_l(t)}{r_l^2} = \frac{U_m(t)}{r_m^2} = \frac{U_s(t)}{r_s^2}$. At l_l , a wetting fluid transfer of dq_w occurs from the large capillary to the medium capillary, whereas the volumetric wetting fluid transferred to the small capillary from the large capillary is dq'_w , shown in Fig. 10(a). The velocity of meniscus in the large capillary is given by,

$$\frac{dl_l}{dt} = U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2}. \quad (C.25)$$

The flow rate in the medium and the small capillaries increases after the fluid transfer at l_l whereas the flow rate of the non-wetting fluid in the large capillary decreases as shown in Fig. 10(c), whose pressure drop is given by,

$$P_{nw}(l_m, t) - P_{nw}(l_l, t) = -\frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2} \right). \quad (C.26)$$

The pressure drop in the medium capillary after l_l is given as,

$$P_w(l_m, t) - P_w(l_l, t) = -\frac{8\mu_w(l_m - l_l)}{r_m^2} \left(U_l(t) \frac{r_m^2}{r_l^2} + \frac{dq_w}{\pi r_m^2} \right), \quad (C.27)$$

where $P_w(l_m, t)$ is the pressure of the wetting fluid in r_m at l_m and $P_w(l_l, t)$ is the pressure of the wetting fluid at l_l in the medium capillary. Then, the velocity of the meniscus in the medium capillary is,

$$\frac{dl_m}{dt} = U_m(t) + \frac{dq_w}{\pi r_m^2}. \quad (C.28)$$

The small capillary is filled with wetting phase from l_l to l_s and the pressure drop in this length is given as,

$$P_w(l_s, t) - P_w(l_l, t) = -\frac{8\mu_w(l_s - l_l)}{r_s^2} \left(U_l(t) \frac{r_s^2}{r_l^2} + \frac{dq'_w}{\pi r_s^2} \right), \quad (C.29)$$

where $P_w(l_l, t)$ is the pressure in the wetting fluid of small capillary at l_l and $P_w(l_s, t)$ is the pressure of the wetting fluid at l_s . Now, the velocity of meniscus in the small capillary is given as,

$$\frac{dl_s}{dt} = U_s(t) + \frac{dq'_w}{\pi r_s^2}. \quad (C.30)$$

At l_m , some of the non-wetting fluid transfers from the medium capillary to the large capillary such that the pressure drop after l_m is same in both the capillaries. Let the non-wetting fluid transfer from the medium capillary to the large capillary be dq_{nw} as shown in Fig. 10(a). After l_m , both the large and the medium capillaries are filled with the non-wetting phase, which gives us

$$\frac{U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2}}{r_l^2} = \frac{U_m(t) + \frac{dq_w}{\pi r_m^2} - \frac{dq_{nw}}{\pi r_m^2}}{r_m^2}. \quad (C.31)$$

The pressure drop of non-wetting fluid from l_m to l_s is given as,

$$P_{nw}(l_s, t) - P_{nw}(l_m, t) = -\frac{8\mu_{nw}(l_s - l_m)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2} + \frac{dq_{nw}}{\pi r_l^2} \right).$$

At l_s , non-wetting fluid transfer occurs from the small capillary to the two larger radii capillaries after which the pressure drop in all

the three capillaries is the same till the exit, which is given by Eq. (C.9). We further know that the capillary pressure is the difference between the non-wetting and wetting phase pressures at the menisci as given by Eq. (C.10). Then, by substituting the capillary pressures and velocities of menisci, and by eliminating the fluid transfers, the equations governing the spontaneous imbibition phenomena in the capillaries are,

$$P_{C_l} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_s)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (C.32)$$

$$P_{C_m} - P_{C_l} = (l_m - l_l) \left(\frac{8\mu_w}{r_m^2} \left(\frac{dl_m}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (C.33)$$

$$P_{C_s} - P_{C_l} = \frac{8\mu_w(l_s - l_l)}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}(l_m - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) - \frac{8\mu_{nw}(l_s - l_m)}{r_l^4 + r_m^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} \right), \quad (C.34)$$

where P_{C_i} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the menisci velocity respectively in a capillary of hydraulic radius r_i . The Eqs. (41)–(43) are written in dimensionless forms as,

$$1 = \frac{x_l + M(1 - x_s)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_m - x_l) \left(\frac{dx_l}{dT} \right) + \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right), \quad (C.35)$$

$$\beta_m - 1 = (x_m - x_l) \left(\frac{1}{\eta_m^2} \left(\frac{dx_m}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right). \quad (C.36)$$

$$\beta_s - 1 = \frac{(x_s - x_l)}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - M(x_m - x_l) \left(\frac{dx_l}{dT} \right) - \frac{M(x_s - x_m)}{1 + \eta_m^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} \right), \quad (C.37)$$

where $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_i}{r_l}$ and $T = t \frac{P_{C_l} r_l^2}{8\mu_w L^2}$.

Now, we consider the case where $l_m > l_s > l_l$ as shown in Fig. 10(d) and (e) and develop governing equations for this case to predict the menisci locations in an interacting three capillary system. The pressure drop in the three capillaries till the first meniscus at l_l is same which is given by Eq. (C.1) and we know that $\frac{U_l(t)}{r_l^2} = \frac{U_m(t)}{r_m^2} = \frac{U_s(t)}{r_s^2}$ is valid till l_l as shown in Fig. 10(e). At l_l , a volumetric wetting fluid transfer of dq_w and dq'_w occurs from the large capillary to the medium and the small capillaries respectively, which decreases the flow rate in the large capillary. The velocity of the first meniscus at l_l is given by Eq. (C.25) and the pressure drop of non-wetting phase in the large capillary after l_l , till l_s is given as,

$$P_{nw}(l_s, t) - P_{nw}(l_l, t) = -\frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2} \right). \quad (C.38)$$

After l_l , the pressure gradient in the small capillary increases as shown in Fig. 10(e) by AC and the pressure drop till l_s is given by Eq. (C.29). The medium capillary having the leading meniscus has a pressure drop from l_l to l_m , which is given by Eq. (C.27). At l_s , a non-wetting fluid transfer of dq'_{nw} occurs from the small capillary to the large capillary such that the pressure drop in both these capillaries remains same till l_m , which is given as,

$$P_{nw}(l_m, t) - P_{nw}(l_s, t) = -\frac{8\mu_{nw}(l_m - l_s)}{r_l^2} \left(U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2} + \frac{dq'_{nw}}{\pi r_l^2} \right). \quad (C.39)$$

The same pressure drop from l_s till l_m in the large and the small capillaries gives the relation $\frac{U_l(t) - \frac{dq_w}{\pi r_l^2} - \frac{dq'_w}{\pi r_l^2} + \frac{dq'_{nw}}{\pi r_l^2}}{r_l^2} = \frac{U_s(t) + \frac{dq_w}{\pi r_s^2} - \frac{dq'_{nw}}{\pi r_s^2}}{r_s^2}$. At l_m , the non-wetting fluid transfers from the medium capillary such that the pressure drop in all the three capillaries remains same till outlet, which is written as,

$$P_o - P_{nw}(l_m, t) = -\frac{8\mu_{nw}(L - l_m)}{r_l^2} U_l(t). \quad (C.40)$$

From the equations of pressure drop in different lengths, by eliminating the velocities $U_l(t)$, $U_m(t)$, $U_s(t)$ and fluid transfers, we obtain the equations governing the flow behavior by using the difference of the non-wetting and wetting phase pressures as the capillary pressure, which is described in Eq. (C.10). The equations governing the spontaneous imbibition for an interacting three capillary system with the large capillary at the center and $l_m > l_s > l_l$ are,

$$P_{C_l} = \frac{8[\mu_w l_l + \mu_{nw}(L - l_m)]}{r_l^4 + r_m^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_m^2 \frac{dl_m}{dt} + r_s^2 \frac{dl_s}{dt} \right) + \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) + \frac{8\mu_{nw}(l_m - l_s)}{r_l^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_s^2 \frac{dl_s}{dt} \right), \quad (C.41)$$

$$P_{C_m} - P_{C_l} = \frac{8\mu_w(l_m - l_l)}{r_m^2} \left(\frac{dl_m}{dt} \right) - \frac{8\mu_{nw}(l_s - l_l)}{r_l^2} \left(\frac{dl_l}{dt} \right) - \frac{8\mu_{nw}(l_m - l_s)}{r_l^4 + r_s^4} \left(r_l^2 \frac{dl_l}{dt} + r_s^2 \frac{dl_s}{dt} \right). \quad (C.42)$$

$$P_{C_s} - P_{C_l} = (l_s - l_l) \left(\frac{8\mu_w}{r_s^2} \left(\frac{dl_s}{dt} \right) - \frac{8\mu_{nw}}{r_l^2} \left(\frac{dl_l}{dt} \right) \right), \quad (C.43)$$

where P_{C_i} and $\frac{dl_i}{dt}$ are the effective capillary pressure and the meniscus velocity respectively in a capillary of hydraulic radius r_i . The Eqs. (47)–(49) are written in dimensionless forms as,

$$1 = \frac{x_l + M(1 - x_m)}{1 + \eta_m^4 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_m^2 \frac{dx_m}{dT} + \eta_s^2 \frac{dx_s}{dT} \right) + M(x_s - x_l) \left(\frac{dx_l}{dT} \right) + \frac{M(x_m - x_s)}{1 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_s^2 \frac{dx_s}{dT} \right), \quad (C.44)$$

$$\beta_m - 1 = \frac{(x_m - x_l)}{\eta_m^2} \left(\frac{dx_m}{dT} \right) - M(x_s - x_l) \left(\frac{dx_l}{dT} \right) - \frac{M(x_m - x_s)}{1 + \eta_s^4} \left(\frac{dx_l}{dT} + \eta_s^2 \frac{dx_s}{dT} \right), \quad (C.45)$$

$$\beta_s - 1 = (x_s - x_l) \left(\frac{1}{\eta_s^2} \left(\frac{dx_s}{dT} \right) - M \left(\frac{dx_l}{dT} \right) \right), \quad (C.46)$$

where $x_i = \frac{l_i}{L}$ for $i = l, m, s$, $M = \frac{\mu_{nw}}{\mu_w}$, $\beta_i = \frac{P_{C_i}}{P_{C_l}}$, $\eta_i = \frac{r_i}{r_l}$ and $T = t \frac{P_{C_l} r_l^2}{8\mu_w L^2}$.

References

- Ashraf, S., Visavale, G., Bahga, S.S., Phirani, J., 2017. Spontaneous imbibition in parallel layers of packed beads. *Eur. Phys. J. E* 40 (4), 39.
- Bartley, J., Ruth, D., 1999. Relative permeability analysis of tube bundle models. *Transp. Porous Media* 36 (2), 161–188.
- Bartley, J., Ruth, D., 2001. Relative permeability analysis of tube bundle models, including capillary pressure. *Transp. Porous Media* 45 (3), 445–478.
- Bico, J., Quéré, D., 2003. Precursors of impregnation. *EPL (Europhys. Lett.)* 61 (3), 348–353.
- Birdsell, D.T., Rajaram, H., Dempsey, D., Viswanathan, H.S., 2015. Hydraulic fracturing fluid migration in the subsurface: a review and expanded modeling results. *Water Resour. Res.* 51 (9), 7159–7188.

- Brackbill, J., Kothe, D.B., Zemach, C., 1992. A continuum method for modeling surface tension. *J. Comput. Phys.* 100 (2), 335–354.
- Brooks, R.H., Corey, A.T., 1964. Hydraulic properties of porous media and their relation to drainage design. *Trans. ASAE* 7 (1), 26–0028.
- Budaraju, A., Phirani, J., Kondaraju, S., Bahga, S.S., 2016. Capillary displacement of viscous liquids in geometries with axial variations. *Langmuir* 32 (41), 10513–10521.
- Cate, D.M., Adkins, J.A., Mettakoonpitak, J., Henry, C.S., 2014. Recent developments in paper-based microfluidic devices. *Anal. Chem.* 87 (1), 19–41.
- Celia, M., Bachu, S., Nordbotten, J., Bandilla, K., 2015. Status of CO₂ storage in deep saline aquifers with emphasis on modeling approaches and practical simulations. *Water Resour. Res.* 51 (9), 6846–6892.
- Corey, A.T., 1954. The interrelation between gas and oil relative permeabilities. *Producers Mon.* 19 (1), 38–41.
- Dahle, H.K., Celia, M.A., Majid Hassanizadeh, S., 2005. Bundle-of-tubes model for calculating dynamic effects in the capillary-pressure-saturation relationship. *Transp. Porous Media* 58 (1–2), 5–22.
- Das, S., Mitra, S.K., 2013. Different regimes in vertical capillary filling. *Phys. Rev. E* 87 (6), 063005.
- Das, S., Waghmare, P.R., Mitra, S.K., 2012. Early regimes of capillary filling. *Phys. Rev. E* 86 (6), 067301.
- Dong, M., Dullien, F.A.L., Zhou, J., 1998. Characterization of waterflood saturation profile histories by the 'complete' capillary number. *Transp. Porous Media* 31 (2), 213–237.
- Dong, M., Dullien, F.A., Dai, L., Li, D., 2005. Immiscible displacement in the interacting capillary bundle model part i. Development of interacting capillary bundle model. *Transp. Porous Media* 59 (1), 1–18.
- Dong, M., Dullien, F.A., Dai, L., Li, D., 2006. Immiscible displacement in the interacting capillary bundle model part ii. Applications of model and comparison of interacting and non-interacting capillary bundle models. *Transp. Porous Media* 63 (2), 289–304.
- Du, N., Fan, J., Wu, H., 2008. Optimum porosity of fibrous porous materials for thermal insulation. *Fibers Polym.* 9 (1), 27–33.
- Elenius, M.T., Gasda, S.E., 2013. Convective mixing in formations with horizontal barriers. *Adv. Water Resour.* 62, 499–510.
- Gauss, C.F., 1877. *Principia generalia theoriae figurae fluidorum in statu aequilibrii*. In: Werke. Springer, pp. 29–77.
- Gorce, J.-B., Hewitt, I., Vella, D., 2016. Capillary imbibition into converging tubes: beating Washburn's law and the optimal imbibition of liquids. *Langmuir*.
- Gunda, N.S.K., Bera, B., Karadimitriou, N.K., Mitra, S.K., Hassanizadeh, S.M., 2011. Reservoir-on-a-chip (ROC): a new paradigm in reservoir engineering. *Lab Chip* 11 (22), 3785–3792.
- Harvie, D.J., Davidson, M., Rudman, M., 2006. An analysis of parasitic current generation in volume of fluid simulations. *Appl. Math. Model.* 30 (10), 1056–1066.
- Hirt, C.W., Nichols, B.D., 1981. Volume of fluid (VOF) method for the dynamics of free boundaries. *J. Comput. Phys.* 39 (1), 201–225.
- Hultmark, M., Aristoff, J.M., Stone, H.A., 2011. The influence of the gas phase on liquid imbibition in capillary tubes. *J. Fluid Mech.* 678, 600–606.
- Li, S., Dong, M., Luo, P., 2017. A crossflow model for an interacting capillary bundle: development and application for waterflooding in tight oil reservoirs. *Chem. Eng. Sci.* 164, 133–147.
- Lucas, R., 1918. Rate of capillary ascension of liquids. *Kolloid Z* 23 (15), 15–22.
- Mullins, B.J., Braddock, R.D., Kasper, G., 2007. Capillarity in fibrous filter media: relationship to filter properties. *Chem. Eng. Sci.* 62 (22), 6191–6198.
- Nekouei, M., Vanapalli, S.A., 2017. Volume-of-fluid simulations in microfluidic t-junction devices: influence of viscosity ratio on droplet size. *Phys. Fluids* 29 (3), 032007.
- Ouali, F.F., McHale, G., Javed, H., Trabi, C., Shirtcliffe, N.J., Newton, M.I., 2013. Wetting considerations in capillary rise and imbibition in closed square tubes and open rectangular cross-section channels. *Microfluid. Nanofluid.* 15 (3), 309–326.
- Pan, Z., Weibel, J.A., Garimella, S.V., 2015. Spurious current suppression in VOF-CSF simulation of slug flow through small channels. *Numer. Heat Transf. Part A: Appl.* 67 (1), 1–12.
- Purcell, W., 1949. Capillary pressures—their measurement using mercury and the calculation of permeability therefrom. *J. Petrol. Technol.* 1 (2), 39–48.
- Quéré, D., 1997. Inertial capillarity. *EPL (Europhys. Lett.)* 39 (5), 533.
- Reyssat, M., Courbin, L., Reyssat, E., Stone, H.A., 2008. Imbibition in geometries with axial variations. *J. Fluid Mech.* 615, 335.
- Sarkar, M., Fan, J., Szeto, Y.-c., Tao, X., 2009. Biomimetics of plant structure in textile fabrics for the improvement of water transport properties. *Textile Res. J.* 79 (7), 657–668.
- Shou, D., Fan, J., Ding, F., 2010. A difference-fractal model for the permeability of fibrous porous media. *Phys. Lett. A* 374 (10), 1201–1204.
- Sun, Y., Kharaghani, A., Tsotsas, E., 2016. Micro-model experiments and pore network simulations of liquid imbibition in porous media. *Chem. Eng. Sci.* 150, 41–53.
- Tanino, Y., Blunt, M.J., 2012. Capillary trapping in sandstones and carbonates: dependence on pore structure. *Water Resour. Res.* 48 (8).
- Ubbink, O., Issa, R., 1999. A method for capturing sharp fluid interfaces on arbitrary meshes. *J. Comput. Phys.* 153 (1), 26–50.
- Unsal, E., Mason, G., Morrow, N.R., Ruth, D.W., 2009. Bubble snap-off and capillary-back pressure during counter-current spontaneous imbibition into model pores. *Langmuir* 25 (6), 3387–3395.

- Waghmare, P.R., Mitra, S.K., 2012. A comprehensive theoretical model of capillary transport in rectangular microchannels. *Microfluid. Nanofluid.* 12 (1–4), 53–63.
- Washburn, E.W., 1921. The dynamics of capillary flow. *Phys. Rev.*
- Westerhof, N., Stergiopoulos, N., Noble, M.I., 2010. Springer Science & Business Media.
- Xiao, B., Fan, J., Ding, F., 2012. Prediction of relative permeability of unsaturated porous media based on fractal theory and monte carlo simulation. *Energy Fuels* 26 (11), 6971–6978.
- Zhang, H., Nikolov, A., Wasan, D., 2014. Enhanced oil recovery (EOR) using nanoparticle dispersions: underlying mechanism and imbibition experiments. *Energy Fuels* 28 (5), 3002–3009.