COL 758: Advanced Algorithms

Lecture 03

Discrete Probability Space

Definition:

A discrete probability space (Ω, P) consists of a countable set Ω and a mapping $\Omega \to [0,1]$.

- Condition: $\displaystyle\sum_{\omega \in \Omega} P(\omega) = 1$

Random Variable

Definition:

A real valued function X defined over a probability space (Ω, P).

- Expected value: $E(X) = \displaystyle\sum_{\omega \in \Omega} X(\omega) \cdot P(\omega)$

Biased Coin Toss:

$$\Omega = \{H, T\}$$

$$P(H) = 0.6 \text{ and } P(T) = 0.4$$

$$X = \begin{cases} 0 & \text{if } \omega = H \\ 1 & \text{if } \omega = T \end{cases}$$

$$E(X) = 0\ (0.6) + \cancel{0}\,1\ (0.4) = 0.4$$

Two Unbiased Coin Tosses:

$$\Omega = \{HH, TT, HT, TH\}$$

$$P(HH) = \cdots = P(TT) = 0.25$$

No of Tails

$$X = \begin{cases} 0 & \text{if } \omega = HH \\ 1 & \text{if } \omega \in \{TH, HT\} \\ 2 & \text{if } \omega = TT \end{cases}$$

$$E(X) = 0\ (0.25) + 1\ (0.5) + 2\ (0.25) = 1$$

Linearity of Expectation

Let Y and $X_1, X_2, \ldots, X_n$ be random variables defined over a probability space (Ω, P) such that

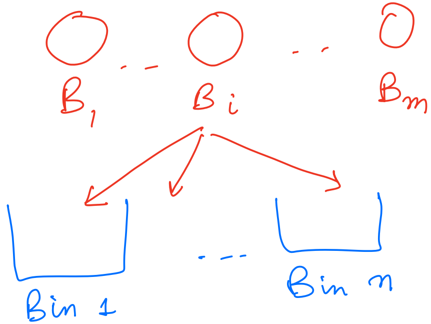
$$Y(\omega) = X_1(\omega) + \cdots + X_n(\omega), \text{ for each } \omega \in \Omega$$

That is, Y is the sum of random variables $X_1, X_2, \ldots, X_n$.

Then, $E(Y) = E(X_1) + E(X_2) + \ldots + E(X_n)$

Proof - homework

Balls Bin Problem



Given: m **balls** and n **bins.** $(m, n \geq 2)$

Each ball is dropped into a bin selected randomly uniformly and independently.

Random Variable X: The number of empty bins.

What is the expected value of X ?

Balls Bin Problem

Given: m **balls** and n **bins.**

Each ball is dropped into a bin selected randomly uniformly and independently.

Random Variable X: The number of empty bins.

What is the expected value of X ?

$\text{Prob}(\text{Ball } 1 \notin \text{Bin } i) = \left(\frac{n-1}{n}\right)$

$\vdots$

$\text{Prob}(\text{Ball } m \notin \text{Bin } i) = \left(\frac{n-1}{n}\right)$

$\Rightarrow \text{Prob}(\text{Bin } i \text{ is empty}) = \left(\frac{n-1}{n}\right)^m$

Solution:

Let $X_i = \begin{cases} 1 & \text{if } i^{th} \text{ bin is empty,} \\ 0 & \text{otherwise .} \end{cases}$

Then, $E(X_i) = P(X_i = 1) = \left(1 - \frac{1}{n}\right)^m$.

Since $X = X_1 + \cdots + X_n$, by Linearity of Expectation we have, $E(X) = \sum_{i=1}^{n} E(X_i) = n\left(1 - \frac{1}{n}\right)^m$

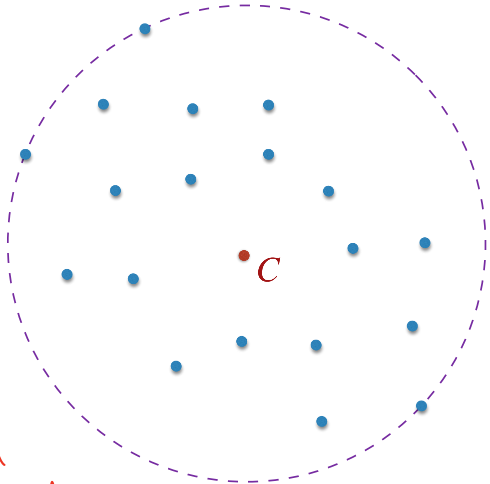
H.W. $\sum_{i=1}^{n} k \cdot \text{Prob}(\text{exactly } k \text{ bins are empty}).$ ← More complicated

Smallest Enclosing Circle

Given: Set of n points in 2D plane - $(x_1, y_1), \ldots, (x_n, y_n)$.

Output: Circle of minimum radius that contains all points.

Or, find a centre $C = (x_0, y_0)$ for which $\max\limits_{1 \le i \le n} \left\{ \sqrt{(x_i - x_0)^2 + (y_i - y_0)^2} \right\}$ is least.



Observation

Either 2 or 3 points will lie on boundary.

$O(n^3)$

Naive Algorithm:

Try all (n choose 3 points) and (n choose 2) points.

Time: $O(n^4)$

verifying a circle $- O(n)$ time

Homework: Think about a natural randomized algorithm!

Smallest Enclosing Circle (Simpler Problem)

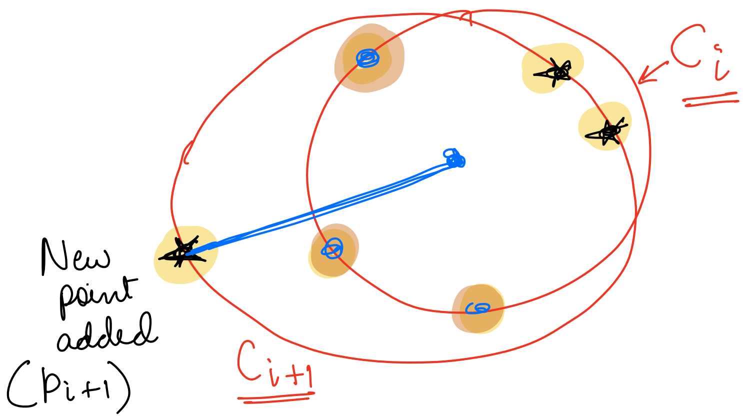
Question: Let $P = (p_1, \ldots, p_n)$ be a random permutation of the n points (x_1, y_1), ..., (x_n, y_n) in 2D plane.

If we incrementally update the 'smallest enclosing circle' by adding one point at a time, then what is the expected number of times we update the circle?

i

C_i

smallest enclosing circle for $p_1 \ldots p_i$

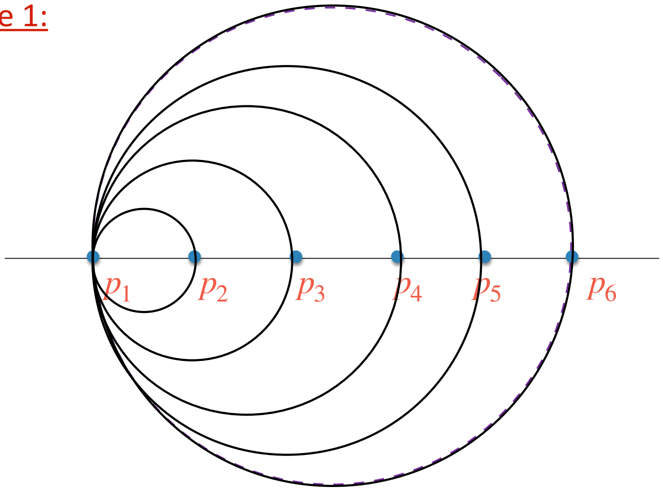


Smallest Enclosing Circle (Simpler Problem)

***Question*:** Let $P = (p_1, \ldots, p_n)$ be a <u>random permutation</u> of the n points (x_1, y_1), ..., (x_n, y_n) in 2D plane.

If we incrementally update the 'smallest enclosing circle' by adding one point at a time, then what is the expected number of times we update the circle?

Case 1:



No of updates = 5

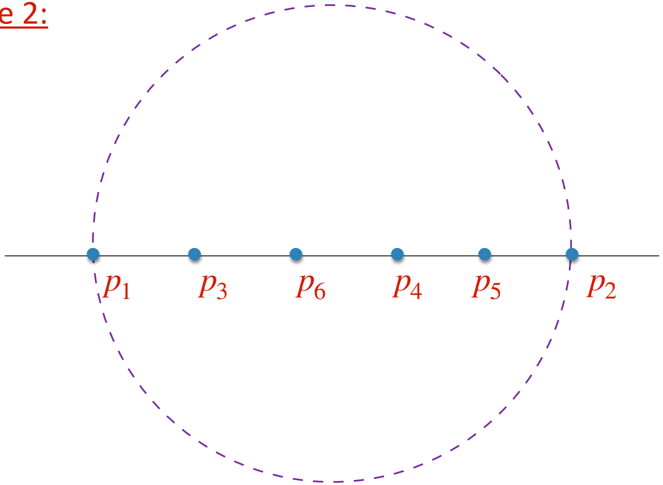
Bad Permutation

Smallest Enclosing Circle (Simpler Problem)

***Question*:** Let $P = (p_1, \ldots, p_n)$ be a random permutation of the n points (x_1, y_1), ..., (x_n, y_n) in 2D plane.

If we incrementally update the 'smallest enclosing circle' by adding one point at a time, then what is the expected number of times we update the circle?

Case 2:



No of updates = 1

Good permutation

Smallest Enclosing Circle (Simpler Problem)

Question: Let $P = (p_1, \ldots, p_n)$ be a random permutation of the n points (x_1, y_1), …, (x_n, y_n) in 2D plane.

If we incrementally update the 'smallest enclosing circle' by adding one point at a time, then what is the expected number of times we update the circle?

Let C_i = smallest circle containing $p_1 \ldots p_i$

$$X_i = \begin{cases} 1 & \text{if circle is updated in } i^{th} \text{ round} \\ 0 & \text{o/w} \end{cases}$$

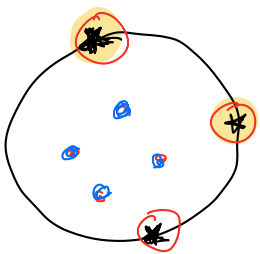
$$\boxed{\begin{aligned} & E(X) \\ & = \sum_{i=1}^{n} E(X_i) \\ & \leq \sum_{i=1}^{n} \frac{3}{i} \\ & = O(\log n) \end{aligned}}$$

CLAIM: $E(X_i) = \text{Prob}(X_i = 1) \leq \frac{3}{i}$

Proof: S = set of any 3 (or 2) points on boundary of C_i

First i ...

C_i



$\text{Prob}(p_i \in S) = \frac{|S|}{i} \leq \frac{3}{i}$

Observation: In i^{th} round circle is updated ONLY IF $p_i \in S$.

$p_1 \quad p_2 \quad p_3 \quad p_i \quad \Big| \quad p_{i+1} \quad p_n$

C_i

$|S| \leq 3$

(S is a function of circle C_i)

Question: Among all these i points, what is probability that at i^{th} position we choose a point from set S?

Note: Point at i^{th} position could have been any of first i points

Smallest Enclosing Circle (An EXPECTED $O(n^3 \log n)$-time algorithm)

Given: Set of n points in 2D plane: $(x_1, y_1), \ldots, (x_n, y_n)$.

Output: Circle of minimum radius that contains all points.

1. Let $P = (p_1, \ldots, p_n)$ be a random permutation of the n points.
2. $C =$ Circle with p_1, p_2 as end-points
3. For ($i = 3$ to n):
4. **If** ($p_i \in C$) **then** DO NOTHING
5. **Else:**

    ForEach ($S \subseteq \{p_1, \ldots, p_{i-1}\}$ of size at most two):

    $C =$ circle passing through $S \cup \{p_i\}$ ← set of size 2 or 3

    If (C encloses $\{p_1, \ldots, p_i\}$) **then** Break

Break inner FOR Loop

Inside For Loop

Smallest Enclosing Circle (Natural Randomized algorithm)

Given: Set of n points in 2D plane: $(x_1, y_1), \ldots, (x_n, y_n)$.

Output: Circle of minimum radius that contains all points.

FindCircle(P, X):

1. Let p be a uniformly random point in set P
2. $C =$ FindCircle($P - \{p\}, X$)
3. **If** $(p \in C)$ **then** Return C

   Else Return FindCircle($P - \{p\}, X \cup \{p\}$)

Homework

- Prove correctness (is it non trivial ?)
- Analyse Time Complexity

Expected Time Complexity