COL 758: Advanced Algorithms

Lecture 04

Randomized Quick Sort

QuickSort(L)

Choose some pivot x from list L

Initialise L_1 , L_2 to be empty lists

For each ($y \in L - \{x\}$):

If ($y < x$) **then** L_1.append(x)

If ($y \geqslant x$) **then** L_2.append(x)

Return QuickSort(L_1) $\circ\, x \,\circ$ QuickSort(L_2)

Randomized Quick Sort

4	3	6	2	11	5	8	7

$\boxed{\text{One time}}$

(Random permutation
computed only once)

5	11	3	8	2	6	4	7

Randomly Permuted List *L*

5	3	2	4	11	8	6	7

pivot — Set L1 = "< 5" — Set L2 = "≥ 5"

1. Divide into smaller subproblems

5	2	3	4	6	7	8	11

pivot — Sorted L1 — Sorted L2

2. Solve the smaller sub-problems recursively

2	3	4	**5**	6	7	8	11

3. Combine

Randomized Quick Sort

Pre-processing (step 0): Randomly permute list L.

QuickSort(L)

Choose pivot x to be first element from list L

Initialise L_1 , L_2 to be empty lists

For each $(y \in L - \{x\})$:

If $(y < x)$ **then** L_1.append(x)

If $(y \geqslant x)$ **then** L_2.append(x)

Return QuickSort$(L_1) \circ x \circ$ QuickSort(L_2)

1. **Divide:** Take first element of list, say x, to be pivot, and divide $L - \{x\}$ into two lists L_1 and L_2 satisfying

$$\text{Elements of } L_1 \ < \ x \ \leqslant \ \text{Elements of } L_2$$

2. **Conquer:** Recursively sort L_1 and L_2

3. **Combine:** Output $(L_1 \cdot x \cdot L_2)$

Claim: For any input list of size n, the expected time taken by quick sort algorithm is $O(n \log n)$.

Randomized Quick Sort

Pre-processing (step 0): Randomly permute list L.

QuickSort(L)

Choose pivot x to be first element from list L

Initialise L_1 , L_2 to be empty lists

For each $(y \in L - \{x\})$:

  If $(y < x)$ **then** L_1.append(x)

  If $(y \geqslant x)$ **then** L_2.append(x)

Return QuickSort(L_1) $\circ\, x \,\circ$ QuickSort(L_2)

Some Notations

e_i : i^{th} smallest element in list L.

So, $[e_1, \ldots, e_n]$ = Sorted L.

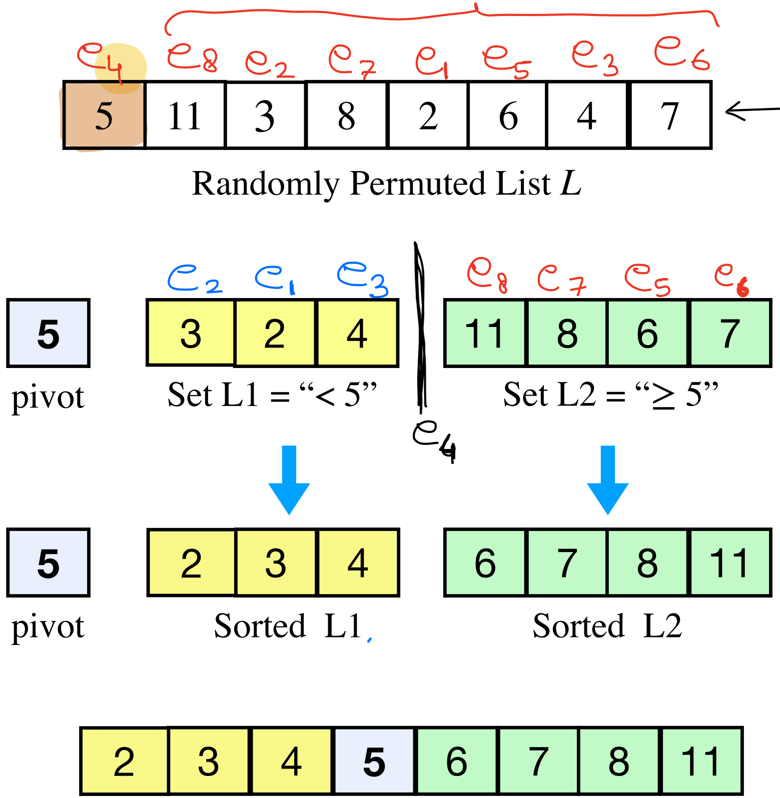
$[e_i, \ldots, e_j]$ = Elements b/w e_i and e_j in sorted order.

Event

$$X_{ij} = \begin{cases} 1 & e_i \text{ is compared to } e_j \\ 0 & \text{otherwise.} \end{cases}$$

Randomized Quick Sort

$$X_{ij} = \begin{cases} 1 & \text{if } e_i, e_j \text{ are compared} \\ 0 & \text{o/w} \end{cases}$$



$X_{4j} = 1, \quad \forall j \neq 4$

Reason: e_4 first elemt of List and compared to everyone

$X_{21}, X_{23} = 1$

$X_{26} = 0$

$X_{16} = 0$

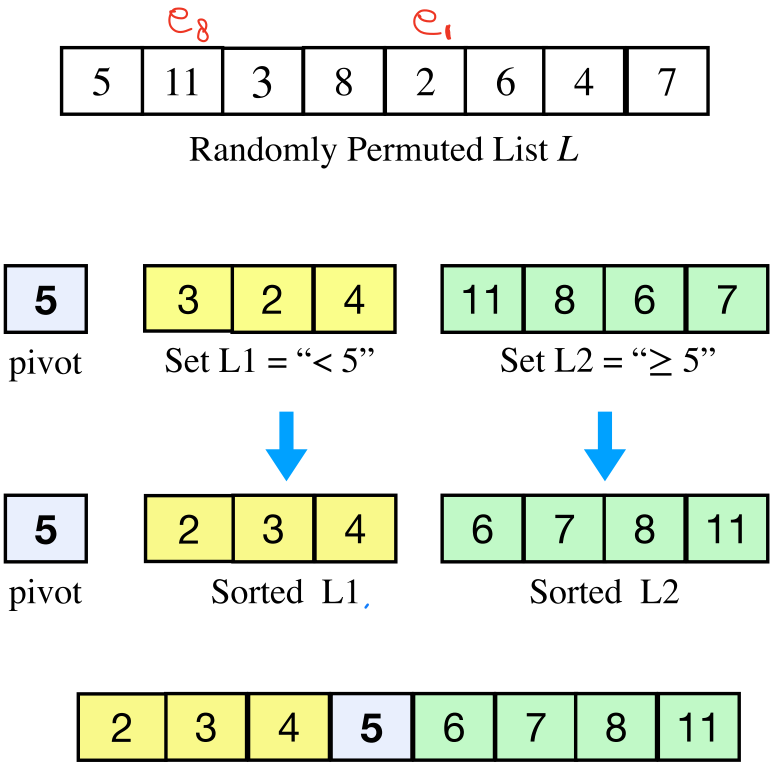
$X_{15} = 0$

$\vdots$

(Total no of comparisons $= \sum_{i<j} X_{ij}$)

Randomized Quick Sort

$$X_{ij} = \begin{cases} 1 & \text{if } e_i \,\&\, e_j \text{ are compared} \\ 0 & \text{o/w} \end{cases}$$



Ques: $\text{Prob}(X_{18} = 1) = \frac{2}{8}$

Ans: $X_{18} = 1$ iff

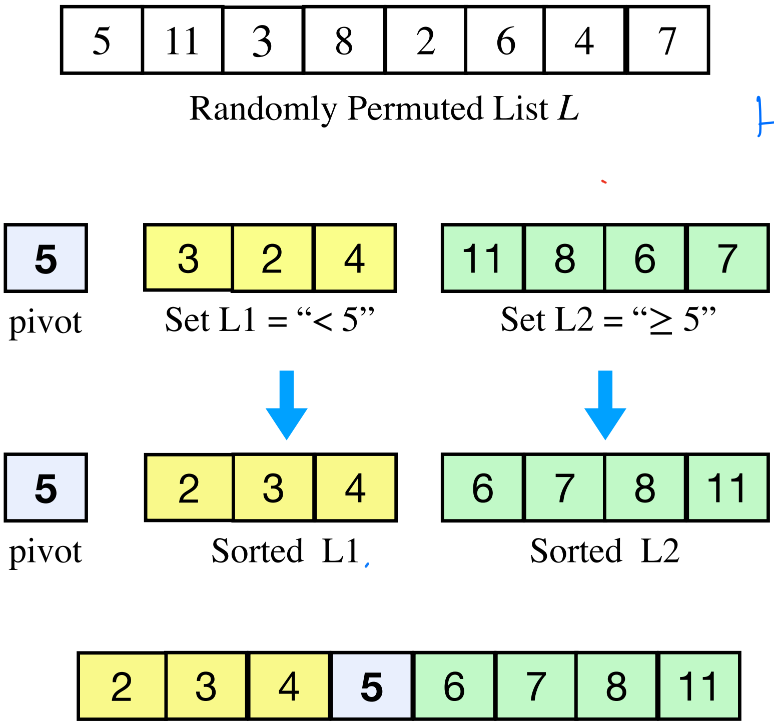
e_1 & e_8 are compared

iff e_1/e_8 is first

element of permutation

Randomized Quick Sort

To Show $\Pr(X_{ij} = 1) = \dfrac{2}{j-i+1}$



Hint: $j-i+1 = \big\| [e_i \cdots e_j] \big\|$

Prob(In the random permutation first element from $[e_i \; e_{i+1} \cdots e_j]$ is either e_i or e_j)

$= \dfrac{2}{j-i+1}$

Randomized Quick Sort

$$X_{ij} = \begin{cases} 1 & e_i \text{ is compared to } e_j \\ 0 & \text{otherwise.} \end{cases}$$

When is $X_{ij} = 1$?

$X_{ij} = 1$ iff the first element from $[e_i, \ldots, e_j]$ to be picked as pivot element is e_i or e_j. (why?) See Next page

So, $E\left(X_{ij}\right) = Prob\left(X_{ij} = 1\right) = \dfrac{2}{j - i + 1}$

CLAIM: $\text{Prob}(X_{ij} = 1) = \frac{1}{j-i+1}$

PROOF SKETCH

In a call of QuickSort(L), suppose $[e_i \dots e_j] \subseteq L$. Then while splitting L into L_1 & L_2,

$[e_i \dots e_j] \subseteq L_1$ iff $e_j < \text{pivot}$

$[e_i \dots e_j] \subseteq L_2$ iff $\text{pivot} < e_i$

$[e_i \dots e_j]$ splits iff $\text{pivot} \in [e_i \dots e_j]$

Note

- once $[e_i \dots e_j]$ splits, we never compare e_i & e_j in future
- while $[e_i \dots e_j]$ is splitting, (i) $\text{pivot} \in [e_i \dots e_j]$
  (ii) e_i & e_j compared iff $\text{pivot} = e_i$ or e_j.

Now, conditioned on fact $\text{pivot} \in [e_i \dots e_j]$,

$\text{pivot} = e_i / e_j$ iff e_i / e_j are first element from $[e_i \dots e_j]$ in the permutation

Probability of this event is $2/|[e_i \dots e_j]| = \frac{2}{j-i+1}$. $\square$

Randomized Quick Sort

Total number of comparisons = $\sum_{i<j} X_{ij}$

Expected time complexity

$$E\left(\sum_{i<j} X_{ij}\right) = \sum_{i<j} E\left(X_{ij}\right) = \sum_{i<j} \frac{2}{j-i+1} = \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} = O(n \log n)$$