COL 758: Advanced Algorithms

Lecture 05

Coin Toss Problem

Given: A fair coin is tossed " $8t$ " times.

Claim: Probability we get <u>less than t Heads</u> $\leq \left(\frac{3}{4}\right)^{8t}$

<u>Homework exercise (Hint):</u>

$$Prob(\text{At most } t \text{ Heads}) = \sum_{i=0}^{t-1} \underbrace{{}^{8t}C_i \left(\frac{1}{2}\right)^{8t}}_{\text{Prob of } i \text{ Heads}} \leq t\ {}^{8t}C_t \left(\frac{1}{2}\right)^{8t}$$

$${}^{8t}C_0 + {}^{8t}C_1 + \dots + {}^{8t}C_{t-1}$$

$$\leq \left({}^{8t}C_t\right)(t)$$

<u>Stirling's Formula:</u> $n! = c\sqrt{2\pi n}\left(\frac{n}{e}\right)^n$, where c lies between 0.5 and 2

Randomized Quick Sort

QuickSort(L)

If $(|L| = 0)$ Return L

Choose ***random*** pivot x from list L

Initialise L_1 , L_2 to be empty lists

For each $(y \in L - \{x\})$:
    If $(y < x)$ **then** L_1.append(x)
    If $(y \geqslant x)$ **then** L_2.append(x)

Remove all copies of Pivot

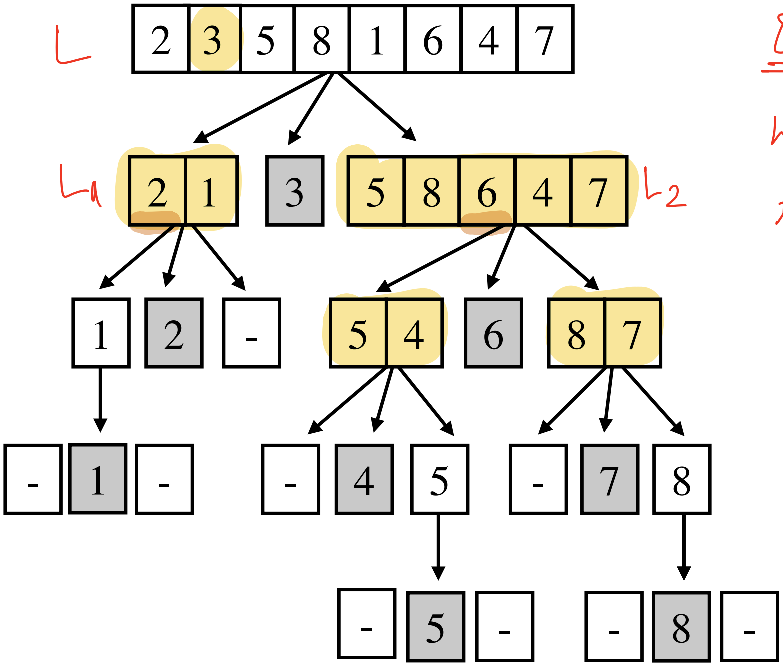
Return QuickSort(L_1) $\circ\, x \,\circ$ QuickSort(L_2)

$L_1 < \text{pivot} \leq L_2$

Theorem: Expected Time Complexity of Quick sort over input of size n is $O(n \log n)$.

How much can runtime of quicksort deviate from expected value?

Recursion Tree



Ques:

What is probability that

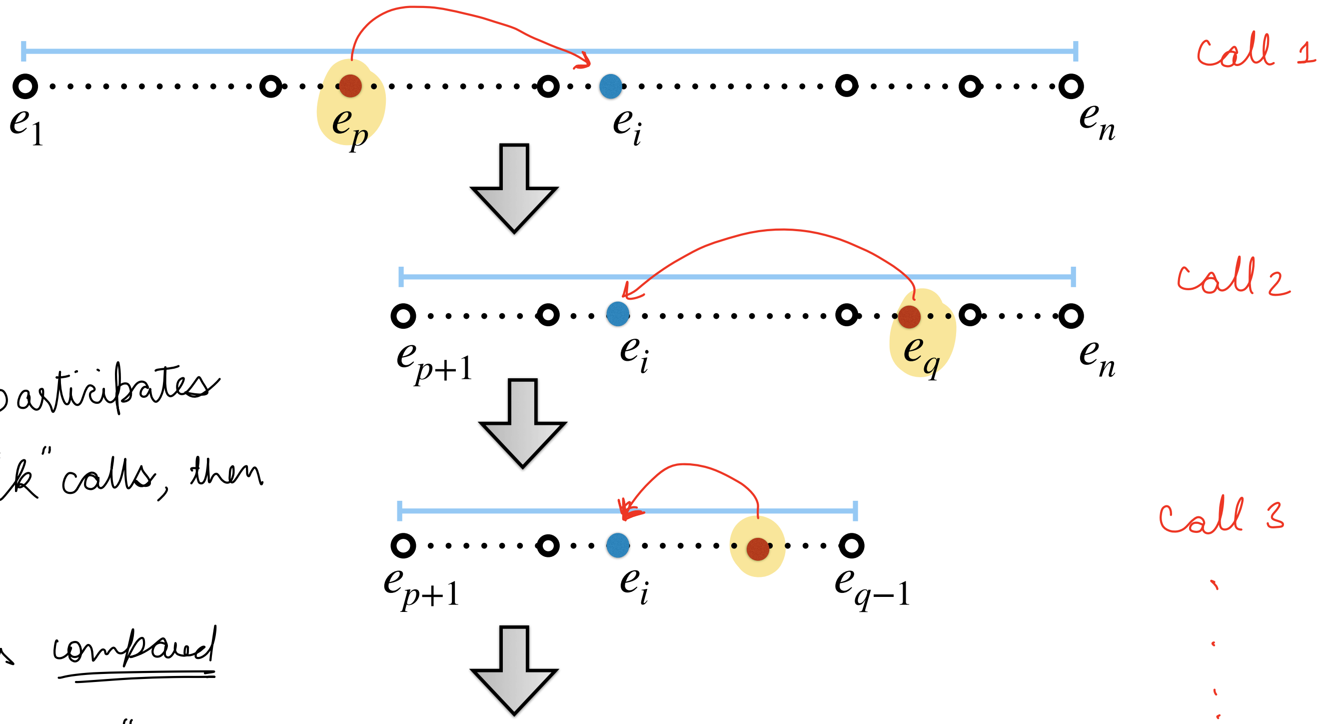
$|L_1|, |L_2| \le \frac{3}{4}|L|$

or

pivot belongs to central half region?

Ans : 1/2

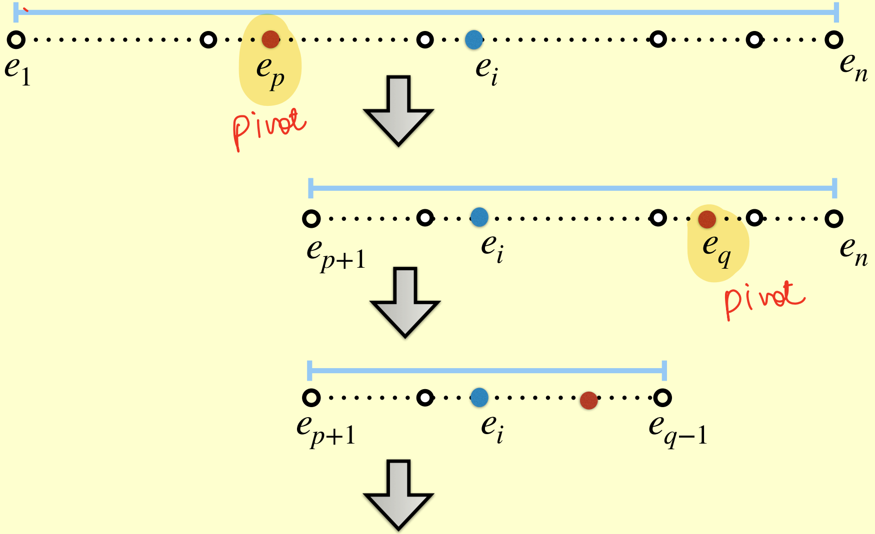
Recursive calls in which e_i participates



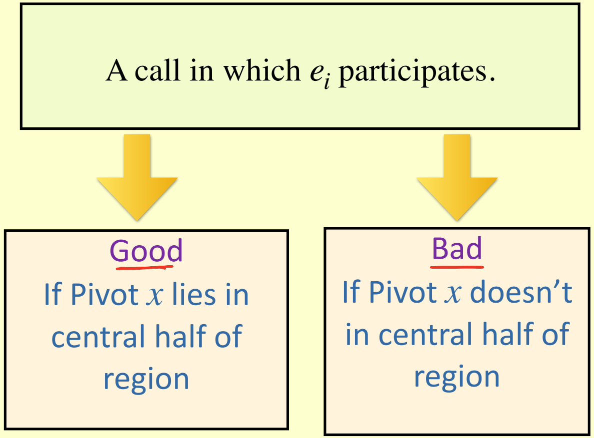
If e_i participates in "k" calls, then

$\Rightarrow$ e_i is compared by pivot "k" no. of times

Recursive calls in which e_i participates



Classification of calls



$|L_1|, |L_2| \leq \frac{3}{4}|L|$

$Prob(\text{good call}) =$ 0.5

Observation: If call is good then, $|L_1|, |L_2| \leq$ 3/4 $|L|$

What is the maximum number of good calls for e_i ?

In a good call

$|L_1|, |L_2| \leq 3/4 \, |L|$

Ans: $\log_{4/3}(n)$

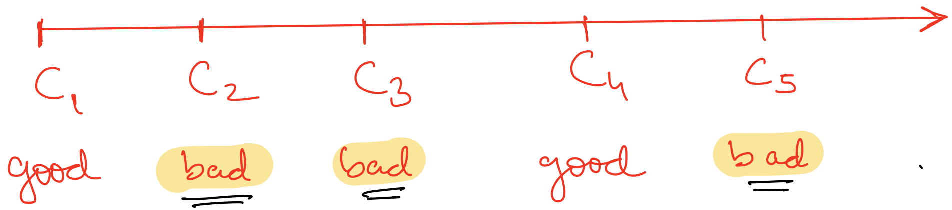
suppose there are "k" good calls,

then the size of list after k good calls will be $\leq (3/4)^k n$

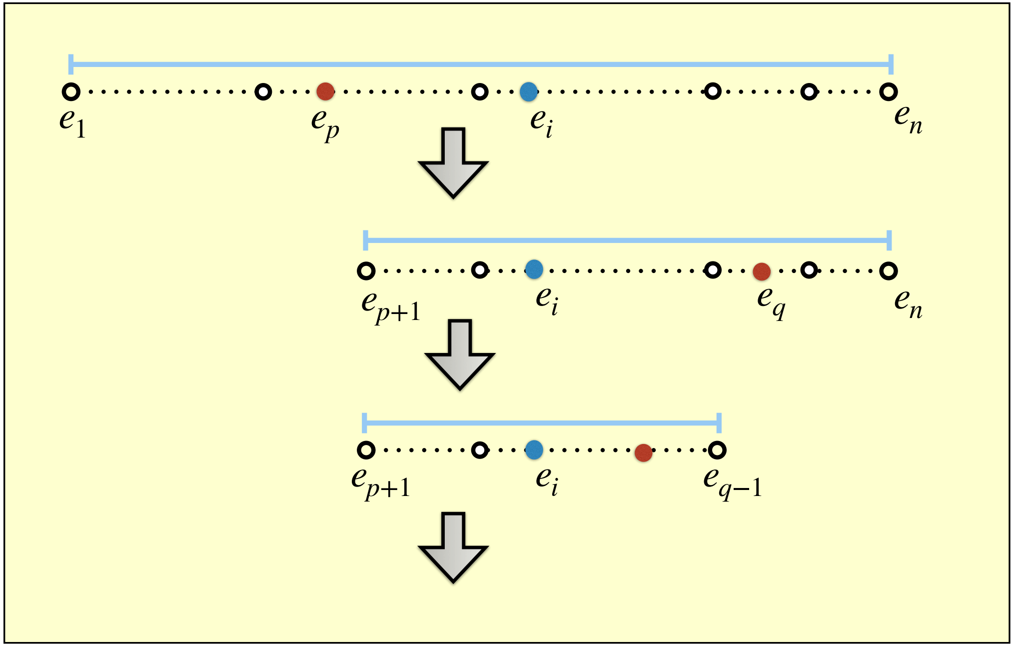
$1 = (3/4)^k n \quad \Rightarrow \quad n = (4/3)^k$

$\Rightarrow \quad k = \log_{4/3}(n)$

Recursive Calls



Recursive calls are Bad/Good independent of each other (why?)



Classification of calls

A call in which e_i participates.

Good

If Pivot x lies in central half of region

Bad

If Pivot x doesn't in central half of region

Question

Prob we take more than $n(8\log_{4/3} n)$ time

after participating in

p_1 Let E_i be the event that e_i ~~participates in at least~~ $8\log_{(4/3)} n$ recursive calls, and

number of good calls are ~~at most~~ less than $\log_{(4/3)} n$.

$$E = \bigcup_{i=1}^{n} E_i$$

$\equiv$ There is some element which after $8\log_{4/3} n$ calls faces $<\log_{4/3} n$ good calls

What is the probability of E_i ? $\leq (3/4)^{8\log_{4/3} n}$

$= (3/4)^{\log_{4/3} n^8}$

$= \frac{1}{n^8}$

$\boxed{p_1 \leq p_2}$

$\text{Prob}(E) \leq \sum_{i=1}^{n} \text{Prob}(E_i) \leq \boxed{\frac{1}{n^7}}$

Analogous

p_2 Prob We toss coin $8t$ no of times (independently), No of heads is $< t$

Ans : $\leq (3/4)^{8t}$

If all elements are in $8\log_{4/3} n$ calls then

time taken $= n * 8\log_{4/3}(n)$

Homework

Ques If you have random permutation in starty

then can you say time taken $\le \boxed{8n \log_{4/3} n}$

with probability $\geq \left(1 - \frac{1}{n^7}\right)$