COL 758: Advanced Algorithms

Lecture 09

Approximate Transitive closure

Given: A directed graph $G=(V,E)$ with n *vertices*.

Find: Estimate size of transitive closure, for each v *in* V.

If $p(v)$ = no. of vertices reachable from v, then

the estimate $\hat{p}(v)$ must lie in $\left[\frac{p(v)}{c}, c \cdot p(v)\right]$

Algorithm $k = \Theta\left(\frac{1}{\varepsilon} \log n\right)$

For ($i = 1$ to k) do:

Assign to each $v \in V$ a uniformly independent random number from [0,1].

For each ($v \in V$): $w_i[v]$ = Label of smallest label vertex reachable from v.

$\Big\}\, O(m + n\log n)$ (why ?)

For each ($v \in V$):

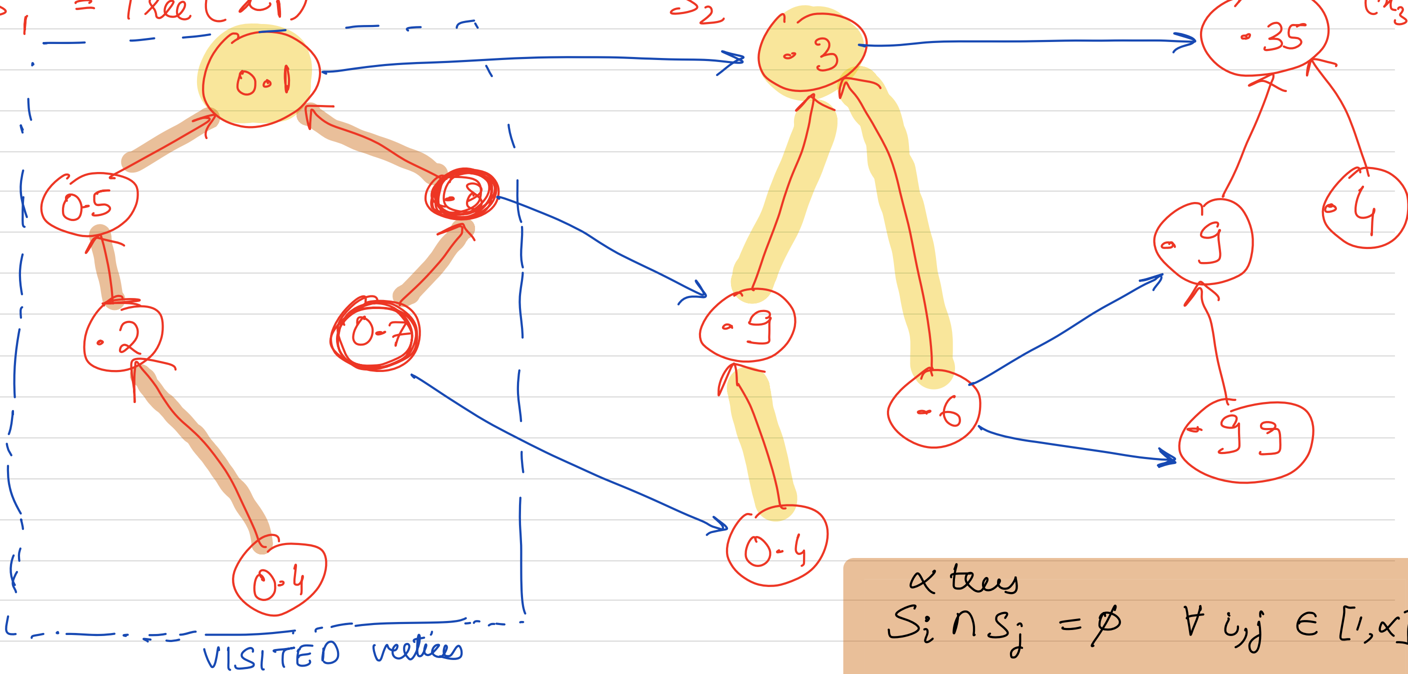
let $w[v] = (k/e)^{th}$ largest element from the set $\{w_1[v], \ldots, w_k[v]\}$.

Output $\hat{p}(v) = \dfrac{1}{w[v]}$

$G = (V, E)$

$$\text{Time} = \sum_{y \in S_1} \text{in-deg}(y) \qquad \text{Time} = \sum_{y \in S_2} \text{in-deg}(y)$$

Set $S_1 = \text{Tree}(x_1)$ $S_2 = \text{Tree}(x_2)$ $S_3 = \text{tree}(x_3)$



α trees

$S_i \cap S_j = \phi \quad \forall\, i,j \in [1, \alpha]$

$\therefore$ Time $= O(m+n)$

$O(n \log n)$

$n = 14$

$[\,0.1 \quad 0.2 \quad 0.3 \quad 0.35 \quad \ldots \quad 0.99\,]$

Boolean Product Witness Matrix (BPWM)

Boolean Matrix Product

Given: Two Boolean square matrices A, B of size n.

Find: The product matrix $C = (A\ B)$.

The matrix C satisfies $C_{ij} = \begin{cases} 1 & \text{if } \sum_{i=1}^{n} A_{ik}B_{kj} > 0 \\ 0 & \text{otherwise} \end{cases}$

Time $= O(n^{\omega})$

Example

$$\underset{C}{\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}} = \underset{A}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}} \times \underset{B}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

Boolean Product Witness Matrix (BPWM)

Best known det - $\Theta(n^3)$

Given: Two Boolean square matrices A, B of size n.

We will provide an $O(n^\omega \log^2 n)$ time randomized algorithm

Find: A matrix W satisfying

$$W_{ij} = \begin{cases} k & \text{if } A_{ik}B_{kj} > 0 \text{ where } k \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

Example

$$\underbrace{\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix}}_{W} = \textbf{BPWM for } \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{A}, \underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{B}$$

1 or 3

Witness at k=1

witness at k=3

Scenario I: Single Witness

Remark: We can't multiply the columns by 1, 20, 300, 4000, & so on because no. of digits will then become $\Omega(n)$. In RAM Model no. of digits must remain $O(\log n)$

Question: Computing witnesses for pairs (i, j) with single witnesses?

$$\overbrace{\begin{array}{c|cccccc} & 1 & 2 & 3 & \cdots & \cdots & n \\ \hline & & & & & & \\ & x & x & x & x & x & x \\ & x & x & x & x & x & x \\ i=3 & 1 & 0 & 1 & 0 & 0 & 0 \\ & x & x & x & x & x & x \\ & x & x & x & x & x & x \\ & x & x & x & x & x & x \end{array}}^{A'} \qquad \overbrace{\begin{array}{cccccc} & & & j=4 & & \\ x & x & x & 0 & x & x \\ x & x & x & 1 & x & x \\ x & x & x & 1 & x & x \\ x & x & x & 0 & x & x \\ x & x & x & 1 & x & x \\ x & x & x & 0 & x & x \end{array}}^{B}$$

We will compute $(A'B)$ in $O(n^\omega)$ time

Claim: We can compute witnesses for all pairs with exactly one witness in $O(n^\omega)$ time.

Issue: Suppose row and column are $\begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ (1 & 0 & 1 & 0 & 0) \end{array} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$.

Then the answer will be $1+3=4$. But 4 is not a witness