COL 758: Advanced Algorithms

Lecture 10

Boolean Product Witness Matrix (BPWM)

Given: Two Boolean square matrices A, B of size n.

Find: A matrix W satisfying

$$W_{ij} = \begin{cases} k & \text{if } A_{ik}B_{kj} > 0 \text{ where } k \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

Example

1 or 3

$$\underset{W}{\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix}} = \textbf{BPWM} \text{ for } \underset{A}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}}, \underset{B}{\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

BPWM Verification

Given: Two Boolean matrices A, B of size n.

Question: Given a matrix W how efficiently can one verify that W = **BPWM**(A, B)?

That is, W should satisfy

$$W_{ij} = \begin{cases} k & \text{if } \; A_{ik}B_{kj} > 0 \;\; \text{where } \; k \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

Verification – Takes $O(n^2)$ time

Scenario I: Single Witness

Question: Computing witnesses for pairs $(i,\ j)$ with single witnesses?

$$
A' \longrightarrow
\begin{array}{c c}
 & \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\
 & \underbrace{\begin{bmatrix}
x & x & x & x & x & x \\
x & x & x & x & x & x \\
1 & 0 & 1 & 0 & 0 & 0 \\
x & x & x & x & x & x \\
x & x & x & x & x & x \\
x & x & x & x & x & x
\end{bmatrix}}_{A} \\
\end{array}
\quad
\begin{array}{l}
\\ \\ i=3 \\ \\ \\
\end{array}
\qquad
\begin{array}{c}
\begin{matrix} & & & j=4 & & \end{matrix} \\
\underbrace{\begin{bmatrix}
x & x & x & 0 & x & x \\
x & x & x & 1 & x & x \\
x & x & x & 1 & x & x \\
x & x & x & 0 & x & x \\
x & x & x & 1 & x & x \\
x & x & x & 0 & x & x
\end{bmatrix}}_{B}
\end{array}
$$

There is a single witness for $(i,j) = (3,4)$.

Claim: We can compute witnesses for all index-pairs $(i,\ j)$ with exactly one witness in $O(n^{\omega})$ time.

$$
R \longrightarrow
\begin{array}{c}
\begin{matrix} 1 & 2 & 3 & 4 & \end{matrix} \\
\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \end{pmatrix}
\end{array}
\qquad
\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\longrightarrow
$$

Ans 5

Wrong ans

Need to verify our answers!

Scenario II: Pairs with exactly " t " Witness

Trick: Sample a set $S \subseteq [1, n]$ where each integer is selected with probability $(1/t)$.

Compute A' only with respect to set S

Scenario II: Pairs with exactly " t " Witness

Suppose (i, j) has $t = 2$ witnesses

Randomly cancel 50% by replacing them with "zero".

$$\begin{array}{cc} & \begin{array}{cccccc} \cancel{1} & 2 & 3 & 4 & 5 & 6 \end{array} \\ i & \begin{bmatrix} & & & & & \\ 1 & 0 & 0 & 1 & 0 & 1 \\ & & & & & \end{bmatrix} \end{array} \qquad \begin{array}{c} j \\ \begin{bmatrix} & 1 & \\ & 1 & \\ & 0 & \\ & 1 & \\ & 0 & \\ & 0 & \end{bmatrix} \end{array}$$

$S \leftarrow$ Randomly selected 50% columns

Expected no of witnesses that remain in $S = 2 \times \frac{1}{2} = 1$.

Scenario II: Pairs with exactly " t " Witness

Suppose a pair (i, j) has "t" witnesses.

Goal – Reduce the number of witnesses from t to 1.

$S \leftarrow$ A random subset of $[1, n]$ where each element selected w.p. $1/t$.

If we select each integer w.p. $1/t$ then

expected no of witnesses selected is $\frac{1}{t} \times t = 1$

Prob of choosing only 1 witness in set S $= \underbrace{\left({}^{t}C_{1}\right)}_{\text{Choices}} \underbrace{\left(\frac{1}{t}\right)}_{\text{Picking the selected witness}} \underbrace{\left(1 - \frac{1}{t}\right)^{t-1}}_{\text{Not picking other } (t-1) \text{ witnesses}} \geq \left(1 - \frac{1}{t}\right)^{t} \geq \frac{1}{4}$

Scenario II: Pairs with exactly " t " Witnesses

Algorithm

For ($r = 1$ to $c \log n$) do:

1. Sample a set S of integers in $[1, n]$ with each integer selected with prob $(1/t)$.
2. Construct a matrix A' such that

$$A'_{ij} = \begin{cases} j & \text{if } A_{ij} \neq 0 \text{ and } j \in S; \\ 0 & \text{otherwise}. \end{cases}$$

3. Compute $X' = AB$.
4. For each pair (i, j): If X'_{ij} is a witness of (i, j) then set $W_{ij} = X'_{ij}$.

$$\text{Prob}\left(\text{algo fails for a pair } (i,j) \text{ with } t \text{ witnesses}\right) \leq \left(\frac{3}{4}\right)^{c \log n} \leq \frac{1}{n^{d+2}}$$

$$\Rightarrow \text{Prob}\left(\text{algo fails for at least 1-pair } (i,j) \text{ with } t \text{ witnesses}\right) \leq \frac{n^2}{n^{d+2}} \leq \frac{1}{n^2}$$

Scenario III: Pairs with Witnesses in range " $[t, 2t]$ "

Consider pair (i,j) with "q" witnesses for some $q \in [t, 2t]$

$S \leftarrow$ subset of columns, each is included with prob $= \frac{1}{t}$

Prob of choosing only 1 witness in set S for (i,j) $= {}^{q}C_1 \, \frac{1}{t} \left(1 - \frac{1}{t}\right)^{q-1} \geq \left(1 - \frac{1}{t}\right)^{q-1}$.

$$\geq \left(1 - \frac{1}{t}\right)^{2t} \geq \frac{1}{16}.$$

Run the outer loop $4 \log_{\frac{16}{15}}(n)$ times, then with probability $\geq 1 - \frac{1}{n^4}$ the algo succeeds, for the pair (i,j)

Randomized Algorithm: BPWM

Initialize W to be zero matrix.

For each ($t \in [1,2,4,8,..,2^{\lceil \log n \rceil}]$) **do**:

Repeat ($c \log n$) **times**:

1. Sample a set S of integers in $[1, n]$ with each integer is selected with probability $(1/t)$.
2. Construct a matrix A' such that

$$A'_{ij} = \begin{cases} j & \text{if } A_{ij} \neq 0 \text{ and } j \in S; \\ 0 & \text{otherwise}. \end{cases}$$

3. Compute $X' = AB$.
4. For each pair (i, j): If X'_{ij} is a witness of (i, j) then set $W_{ij} = X'_{ij}$.

This loop will handle all pairs (i,j) which have witnesses in range $[t, 2t]$.