

COL 758: Advanced Algorithms

Lecture 10

Boolean Product Witness Matrix (BPWM)

Given: Two Boolean square matrices A , B of size n .

Find: A matrix W satisfying

$$W_{ij} = \begin{cases} k & \text{if } A_{ik}B_{kj} > 0 \text{ where } k \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

Example

$1 \ \& \ 3$ — $\begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 4 \\ 1 & 3 & 3 & 2 \\ 1 & 1 & 0 & 0 \end{bmatrix} = \text{BPWM for } \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

W A B

BPWM Verification

Given: Two Boolean matrices A, B of size n .

Question: Given a matrix W how efficiently can one verify that $W = \text{BPWM}(A, B)$?

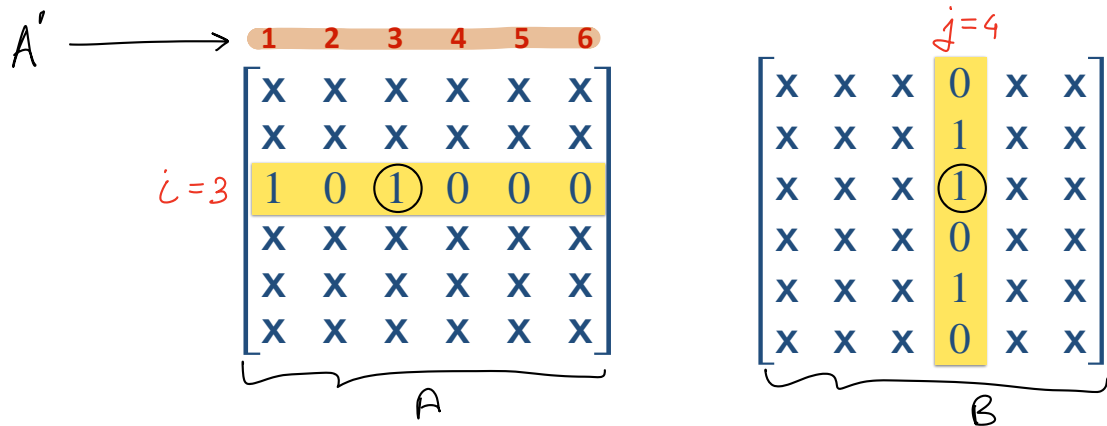
That is, W should satisfy

$$W_{ij} = \begin{cases} k & \text{if } A_{ik}B_{kj} > 0 \text{ where } k \in [1, n] \\ 0 & \text{otherwise} \end{cases}$$

Verification - Takes $O(n^2)$ time

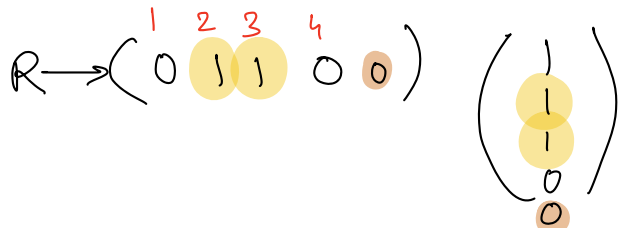
Scenario I: Single Witness

Question: Computing witnesses for pairs (i, j) with single witnesses?



There is a single witness for $(i, j) = (3, 4)$.

Claim: We can compute witnesses for all index-pairs (i, j) with exactly one witness in $O(n^w)$ time.



$\rightarrow$ Ans 5
Wrong ans

} Need to verify our answers!

Scenario II: Pairs with exactly “ t ” Witness

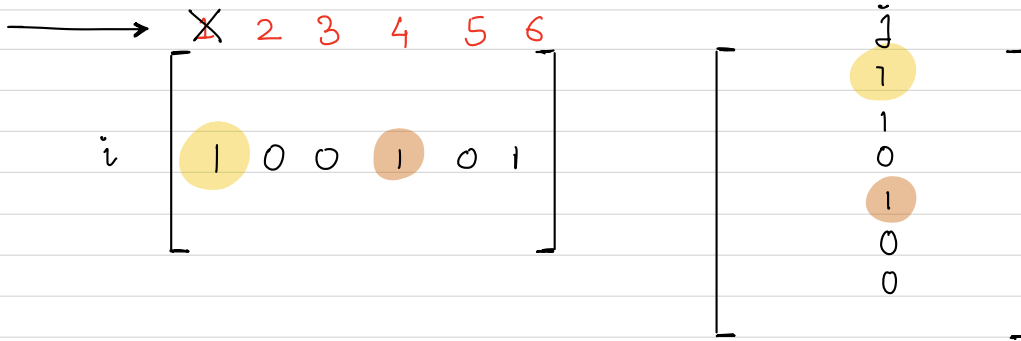
Trick: Sample a set $S \subseteq [1, n]$ where each integer is selected with probability $(1/t)$.

Compute A' only with respect to set S

Scenario II: Pairs with exactly " t " Witness

Suppose (i,j) has $t=2$ witnesses

Randomly
cancel
50% by
replacing
them
with "zero".



$S \leftarrow$ Randomly selected 50% columns

Expected no of witnesses that remain in $S = 2 \times \frac{1}{2} = 1.$

Scenario II: Pairs with exactly " t " Witness

Suppose a pair (i, j) has " t " witnesses.

Goal - Reduce the number of witnesses from t to 1.

$S \leftarrow$ a random subset of $[1, n]$ where each element selected w.p. $\frac{1}{t}$.

If we select each integer w.p. $\frac{1}{t}$ then

expected no of witnesses selected is $\frac{1}{t} \times t = 1$

Prob of choosing only 1 witness in set S =
$$\underbrace{\binom{t}{1}}_{\text{choices}} \underbrace{\left(\frac{1}{t}\right)}_{\text{Picking the selected witness}} \underbrace{\left(1 - \frac{1}{t}\right)^{t-1}}_{\text{Not picking other } (t-1) \text{ witnesses}} \geq \left(1 - \frac{1}{t}\right)^t \geq \frac{1}{4}$$

Scenario II: Pairs with exactly “ t ” Witnesses

Algorithm

For ($r = 1$ to $c \log n$) do:

1. Sample a set S of integers in $[1, n]$ with each integer selected with prob $(1/t)$.
2. Construct a matrix A' such that

$$A'_{ij} = \begin{cases} j & \text{if } A_{ij} \neq 0 \text{ and } j \in S; \\ 0 & \text{otherwise.} \end{cases}$$

3. Compute $X' = AB$.
4. For each pair (i, j) : If X'_{ij} is a witness of (i, j) then set $W_{ij} = X'_{ij}$.

$$\text{Prob}(\text{algo fails for a pair } (i, j) \text{ with } t \text{ witnesses}) \leq \left(\frac{3}{4}\right)^{c \log n} \leq \frac{1}{n^{d+2}}$$

$$\Rightarrow \text{Prob}(\text{algo fails for at least 1-pair } (i, j) \text{ with } t \text{ witnesses}) \leq \frac{n^2}{n^{d+2}} \leq \frac{1}{n^2}$$

Scenario III: Pairs with Witnesses in range " $[t, 2t]$ "

Consider pair (i, j) with " q " witnesses for some $q \in [t, 2t]$

$S \leftarrow$ subset of columns, each is included with prob = $\frac{1}{t}$

$$\begin{aligned} \text{Prob of choosing only 1 witness} &= {}^q C_1 \frac{1}{t} \left(1 - \frac{1}{t}\right)^{q-1} \geq \left(1 - \frac{1}{t}\right)^{q-1} \\ \text{in set } S \text{ for } (i, j) &\geq \left(1 - \frac{1}{t}\right)^{2t} \geq \frac{1}{16} \end{aligned}$$

Run the outer loop $4 \log_{\frac{16}{15}}(n)$ times, then with probability $\geq 1 - \frac{1}{n^4}$

the algo succeeds, for the pair (i, j)

Randomized Algorithm: BPWM

Initialize W to be zero matrix.

For each ($t \in [1, 2, 4, 8, \dots, 2^{\lceil \log n \rceil}]$) **do:**

Repeat ($c \log n$) **times:**

1. Sample a set S of integers in $[1, n]$ with each integer is selected with probability $(1/t)$.

2. Construct a matrix A' such that

$$A'_{ij} = \begin{cases} j & \text{if } A_{ij} \neq 0 \text{ and } j \in S; \\ 0 & \text{otherwise.} \end{cases}$$

3. Compute $X' = AB$.

4. For each pair (i, j) : If X'_{ij} is a witness of (i, j) then set $W_{ij} = X'_{ij}$.

This loop will handle all pairs (i, j) which have witnesses in range $[t, 2t]$.