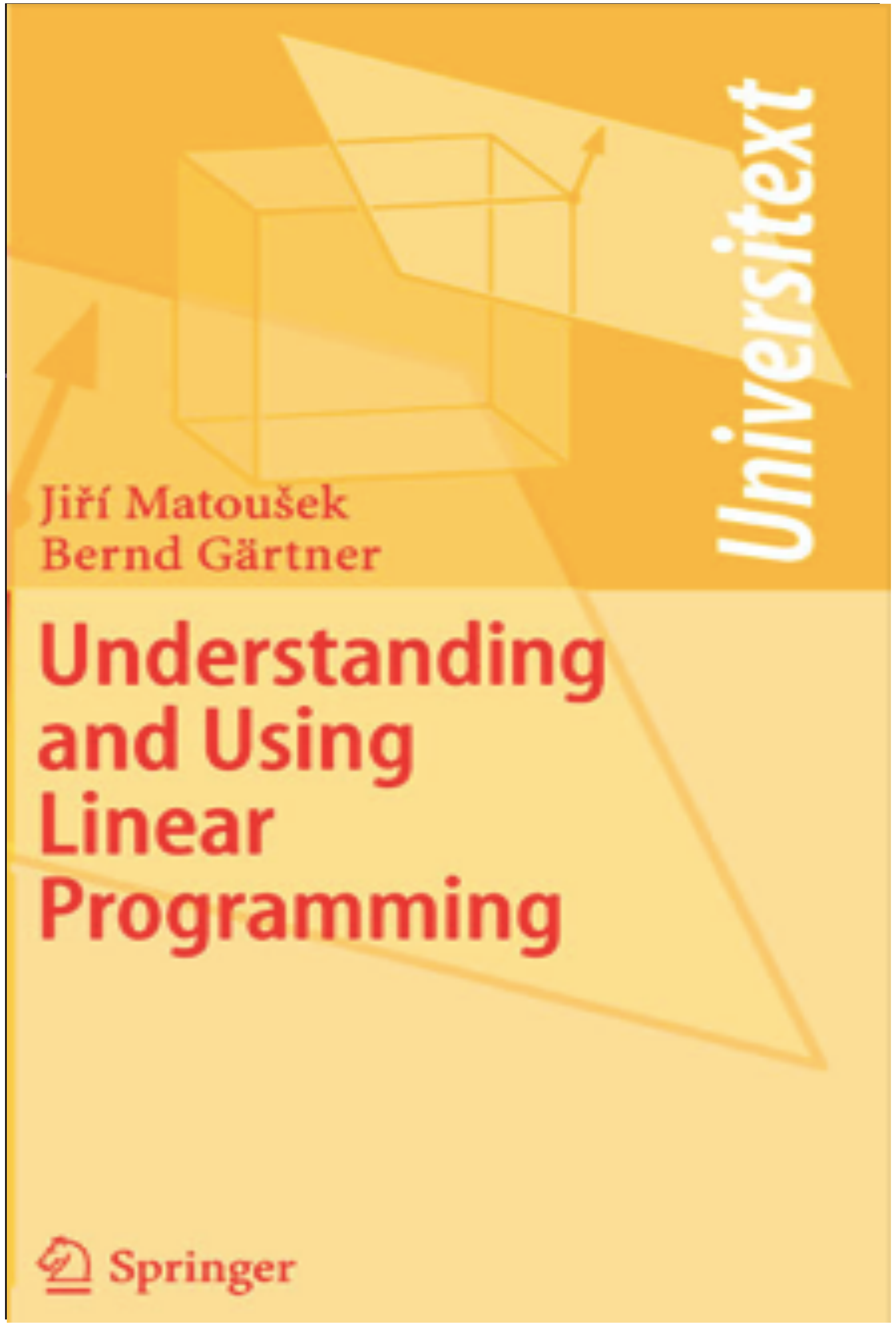
COL 758: Advanced Algorithms

Lecture 19

Reference Book



Understanding and Using Linear Programming
Jiří Matoušek, Bernd Gärtner

Linear Programming

— Feasibility

— Optimization

LP (Feasibility)

Given: List of inequalities, we have n variables and m inequalities.

Find: A feasible solution $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ to all m inequalities, or a proof that no solution exists.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq & b_1 \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq & b_m \end{array} \qquad \forall i, j, \quad a_{ij}, b_j \in \mathbb{Q}$$

"Handling $\leq$": Do negation! Eg. $2x + 3y - z \leq 5 \quad \Leftrightarrow \quad -2x - 3y + z \geq -5$

"Handling $=$": Add both inequalities! Eg. $2x + 3y - z = 5 \quad \Leftrightarrow \quad 2x + 3y - z \geq 5$

$\& \quad -2x - 3y + z \geq -5$

LP (Feasibility)

Given: List of inequalities, we have n variables and m inequalities.

Find: A feasible solution $(r_1, \ldots, r_n) \in \mathbb{Q}^n$ to all m inequalities, or a proof that no solution exists.

$$\begin{bmatrix} a_{11} & a_{12} & & a_{1n} \\ \vdots & & & \\ \vdots & & & \\ \vdots & & & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ \vdots \\ x_n \end{bmatrix} \geq \begin{bmatrix} b_1 \\ \vdots \\ \vdots \\ \vdots \\ b_m \end{bmatrix}$$

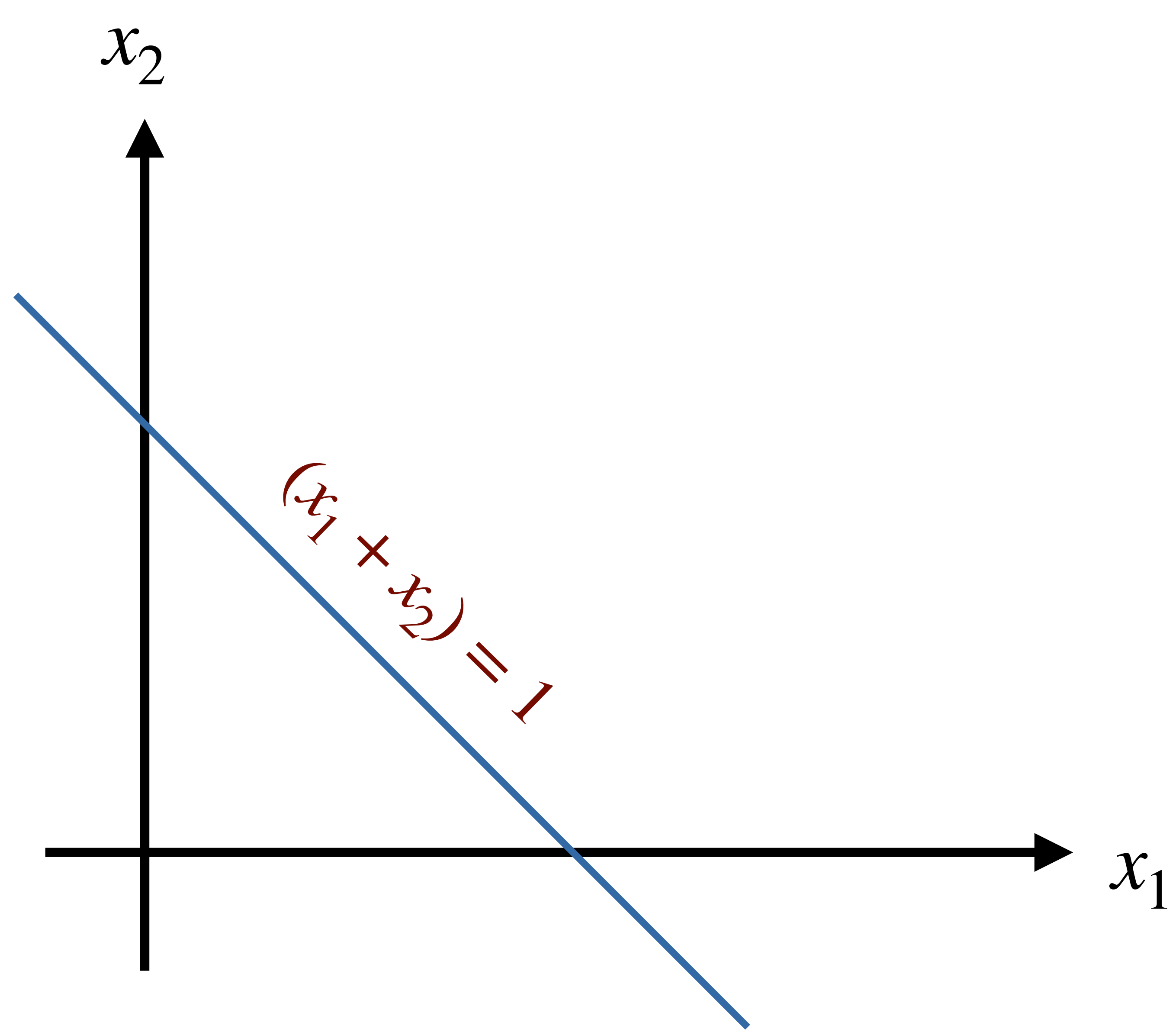
Short form:

$$Ax \geq b$$

Hyperplane

A hyperplane in $\mathbb{R}^n$ is $(n-1)$ dimensional space of the form

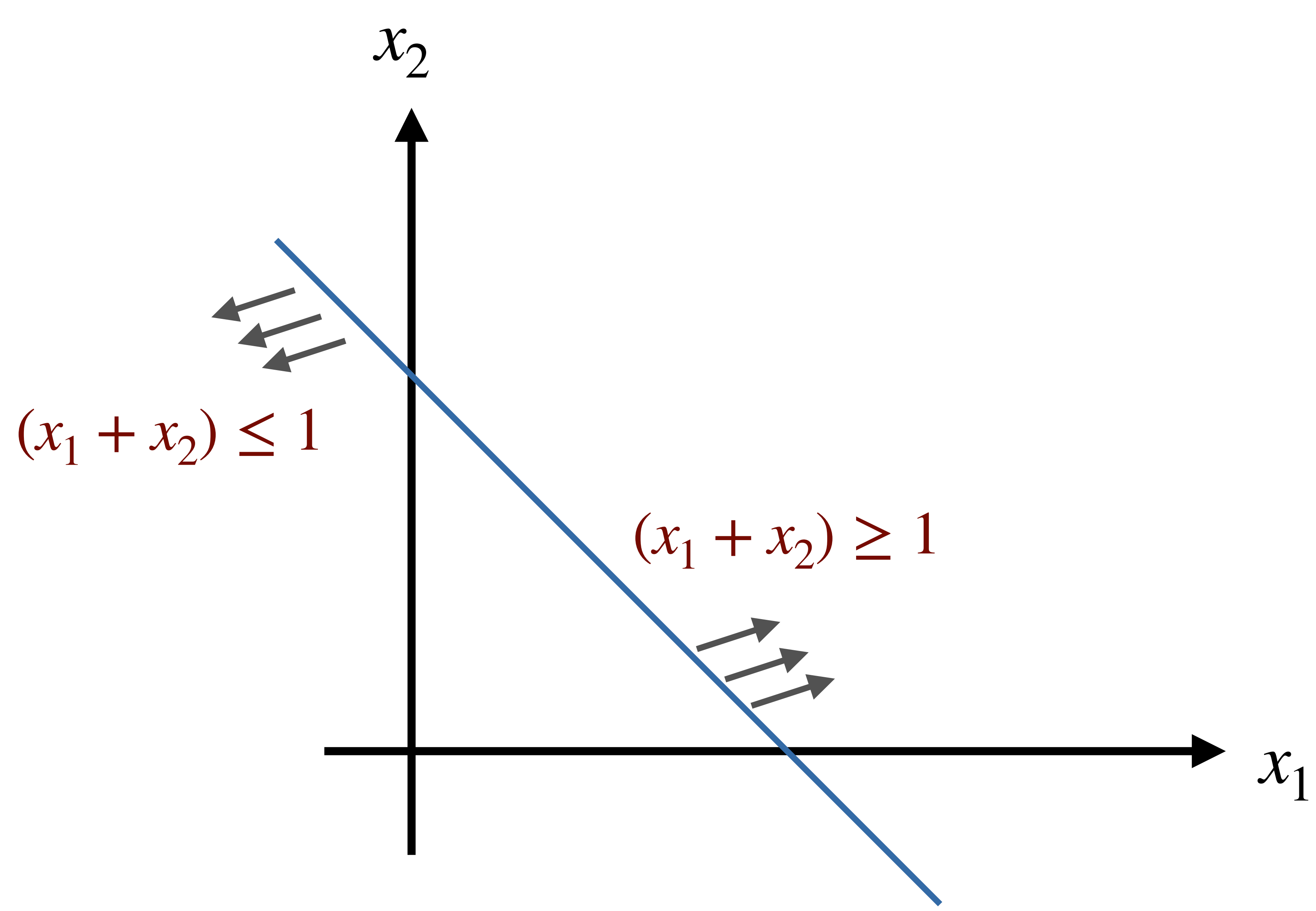
$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n) = b$$



Half-space

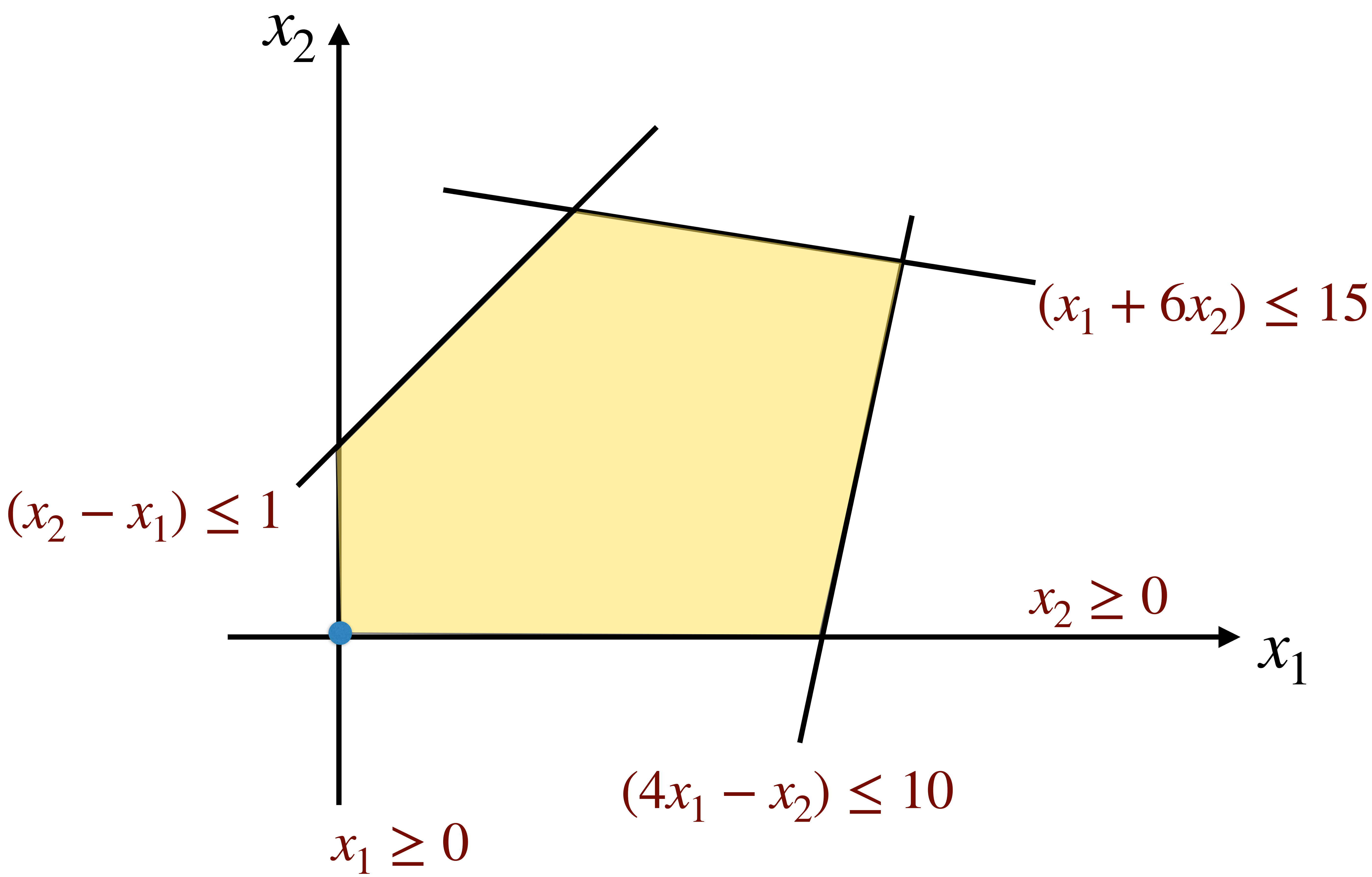
A half-spaces in $\mathbb{R}^n$ has the form

$$(a_1x_1 + a_2x_2 + \ldots + a_nx_n) \geq b$$



LP (Feasibility) — Geometric Interpretation

- Intersection of m half-spaces



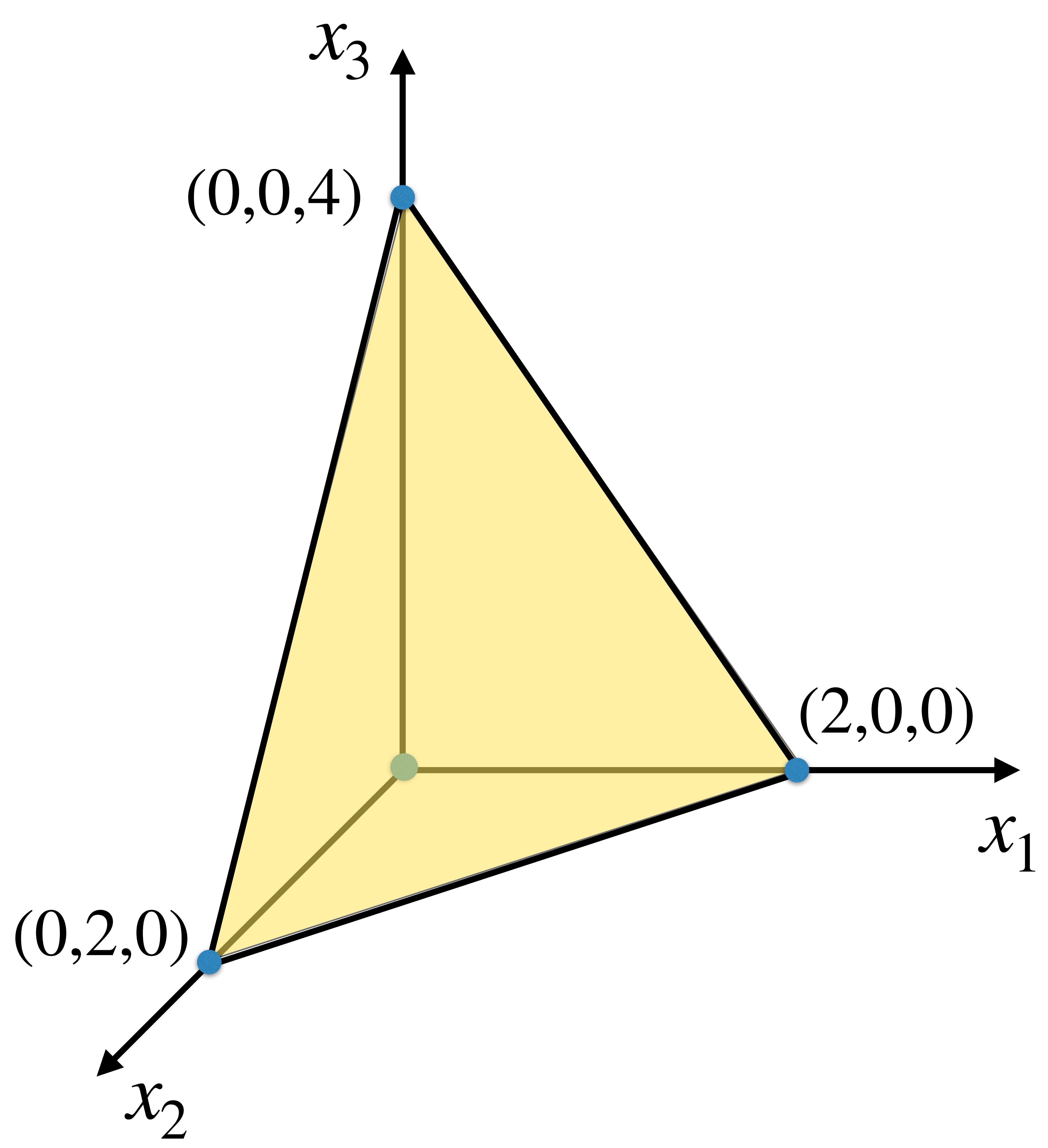
Intersection of five half-spaces in $\mathbb{R}^2$

<u>Polytope</u> formed by intersection of m half-spaces can be:

- Empty
- Unbounded
- Bounded, and non-empty

LP (Feasibility) — Geometric Interpretation

- Intersection of m half-spaces



$$(2x_1 + 2x_2 + x_3) \leq 4$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

$$x_3 \geq 0$$

Polytope is a pyramid with corners

(0,0,0), (2,0,0), (0,2,0), (0,0,4)

LP (Optimization)

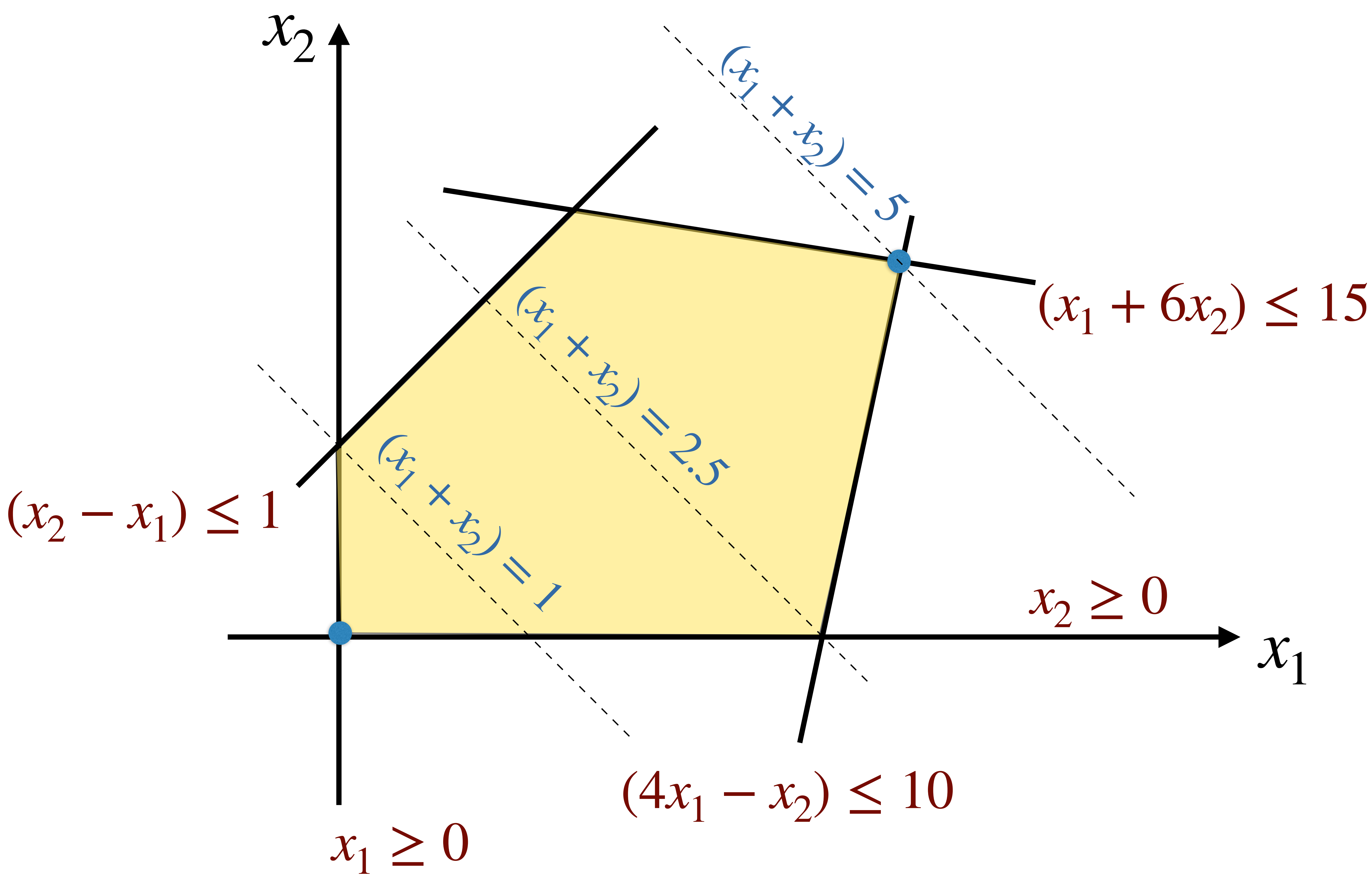
Given: m linear inequalities in n variables, and a vector $c = (c_1, \ldots, c_n) \in \mathbb{Q}^n$.

Find: A feasible solution $x \in \mathbb{Q}^n$ to all m inequalities (if it exists), that maximizes $(c^T x)$.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \ &\geq \ b_1 \\ &\vdots \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \ &\geq \ b_m \end{aligned}$$

LP (Optimization) — Geometric Interpretation

- Intersection of m half-spaces



Objective: max $(x_1 + x_2)$

The hyperplanes in the set

$$\{x_1 + x_2 = \beta \mid \beta \in \mathbb{Q}\}$$

are all parallel to each other.

Among all hyperplanes that intersect the feasible-region the one with the largest β **contains a corner/extreme point** of polytope.

LP (Optimization)

Given: m linear inequalities in n variables, and a vector $c = (c_1, \ldots, c_n) \in \mathbb{Q}^n$.

Find: A feasible solution $x \in \mathbb{Q}^n$ to all m inequalities (if it exists), that maximizes $(c^T x)$.

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What is known?

We can solve feasibility/infeasibility, as well as optimization questions in polynomial time.

LP (Feasibility)

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$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & \geq & b_1 \\ & \vdots & \\ & \vdots & \\ & \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & \geq & b_m \end{array}$$

<u>Aim:</u> Proof or certificate of Feasibility/Infeasibility

We will see first a doubly-exponential time proof!

Examples of some Infeasible LP

Ques: If following system feasible?

$$\begin{aligned} 2x + 3y &\geq 4 \\ -3x + y &\geq -5 \\ x - 4y &\geq 2 \end{aligned}$$

Ans: No, as summing all the inequalities gives $0 \geq 1$.

Ques: If following system feasible?

$$\begin{aligned} 2x - 4y &\geq 5 \\ -x + 2y &\geq -2 \end{aligned}$$

Ans: No, as the sum of following inequalities

$$\left(2x - 4y \geq 5\right) \times 1$$
$$\left(-x + 2y \geq -2\right) \times 2$$

gives $0 \geq 1$.

Infeasibility conditions for an LP

A simple condition of infeasibility: Summing over the equations gives $0 \geq 1$.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\
&\vdots \\
&\vdots \\
\underbrace{a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n}_{0} &\geq \underbrace{b_m}_{\geq 1}
\end{aligned}
$$

Another condition of infeasibility: There exists $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ satisfying:

$$
\begin{aligned}
\lambda_1\left(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \right) &\geq \lambda_1\left(b_1 \right) \\
&\vdots \\
&\vdots \\
\underbrace{\lambda_m\left(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \right)}_{0} &\geq \underbrace{\lambda_m\left(b_m \right)}_{\geq 1}
\end{aligned}
$$

Theorem: Farka's Lemma (LP Duality)

A system of m inequalities of form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

is infeasible **iff** $\exists\ \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ satisfying:

$$\lambda_1\left(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \right) \geq \lambda_1\left(b_1 \right)$$
$$\vdots$$
$$\lambda_m\left(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \right) \geq \lambda_m\left(b_m \right)$$

$$0 \qquad \geq \qquad 1$$

Proof of Feasibility:

An n-dim vector $(x_1, \ldots, x_n)$ satisfying all m inequalities.

Proof of Infeasibility

An m-dim vector $(\lambda_1, \ldots, \lambda_m)$ that on multiplying by m inequalities and summing gives $0 \geq 1$.