

COL 758: Advanced Algorithms

Lecture 20

Farka's Lemma (LP Duality)

A system of m inequalities of form

$$\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \end{array}$$

is infeasible **iff** $\exists \lambda_1, \lambda_2, \dots, \lambda_m \geq 0$ satisfying:

$$\begin{array}{c} \lambda_1 \left(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \right) \geq \lambda_1 \left(b_1 \right) \\ \vdots \\ \lambda_m \left(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \right) \geq \lambda_m \left(b_m \right) \\ \hline \qquad \qquad \qquad 0 \qquad \qquad \qquad \geq \qquad 1 \end{array}$$

Proof of Feasibility:

An n -dim vector $(x_1, \dots, x_n)$ satisfying all m inequalities.

Proof of Infeasibility

An m -dim vector $(\lambda_1, \dots, \lambda_m)$ that on multiplying by m inequalities and summing gives $0 \geq 1$.

Farka's Lemma Proof by **Fourier-Motzkin elimination** (An Example)

Input system:

$$\begin{aligned}x - 2y + z &\geq 3 \\2x - 5y + 4z &\geq 10 \\-3x + 3y - 3z &\geq 12 \\-2x + 14y - 4z &\geq -4 \\-10y + z &\geq -15\end{aligned}$$



$$\begin{aligned}x - 2y + z &\geq 3 \\x - 2.5y + 2z &\geq 5 \\-x + y - z &\geq 4 \\-x + 7y - 2z &\geq -2 \\-10y + z &\geq -15\end{aligned}$$



$$\begin{aligned}x &\geq 2y - z + 3 \\x &\geq 2.5y - 2z + 5 \\y - z - 4 &\geq x \\7y - 2z + 2 &\geq x \\-10y + z &\geq -15\end{aligned}$$



$$\begin{aligned}\min(y - z - 4, 7y - 2z + 2) &\geq \max(2y - z + 3, 2.5y - 2z + 5) \\-10y + z &\geq -15\end{aligned}$$

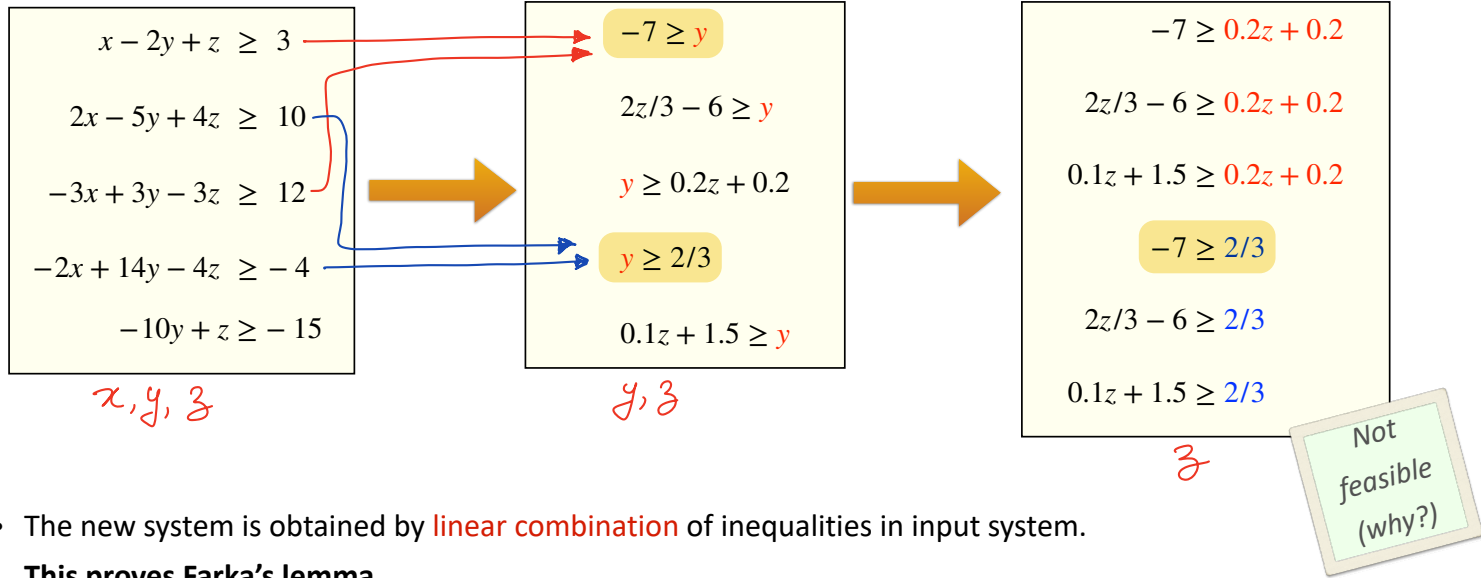


$$\begin{aligned}y - z - 4 &\geq 2y - z + 3 \\y - z - 4 &\geq 2.5y - 2z + 5 \\7y - 2z + 2 &\geq 2y - z + 3 \\7y - 2z + 2 &\geq 2.5y - 2z + 5 \\-10y + z &\geq -15\end{aligned}$$

- The *input system* is feasible **iff** *new system* is feasible.
- The *new system* is obtained by **linear combination** of inequalities in *input system*.

Farka's Lemma Proof by **Fourier-Motzkin elimination** (An Example)

Input



- The new system is obtained by **linear combination** of inequalities in input system.
This proves Farka's lemma.
- On eliminating one variable, new system may have $\Omega(m^2)$ inequalities.
Thus, time complexity is $O(m^{2^n})$.

Farkas's lemma Proof.

m inequalities
 n variables
 $x_1, \dots, x_n$

system 1

$O(m^2)$ inequalities
 $(n-1)$ variables
 $x_1, \dots, x_{n-1}$

system 2

Conversion from system 1 to system 2:

- Write inequalities containing x_n as

$$\underbrace{F(x_1, \dots, x_{n-1})}_{\text{let these be } k \text{ in count}} \geq x_n \quad \text{or} \quad \underbrace{x_n}_{\text{let these be } q \text{ in count}} \geq F(x_1, \dots, x_{n-1}).$$

- Create $k+q$ new inequalities by eliminating x_n .

Claim 1

If system 1 is feasible $\rightarrow$ system 2 is feasible

Proof:

Let $(x_1, \dots, x_n)$ be a solⁿ to system 1. Then, $(x_1, \dots, x_{n-1})$ will be a solⁿ to system 2. Because, inequalities in system 2 are obtained by combination of inequalities in system 1.

Claim 2

If system 2 is feasible $\rightarrow$ system 1 is feasible

Proof:

Let $(x_1, \dots, x_{n-1})$ be a solⁿ to system 2.

Recall: While eliminating x_n we obtained upper & lower bounds on x_n as

$$\min \{ \text{\textcolor{red}{- } } k \text{ inequalities} \} \geq x_n \geq \max \{ \text{\textcolor{red}{- } } q \text{ inequalities} \}$$

Blog the new system is satisfiable, we can find an appropriate value of x_n .

Theorem

m inequalities
 n variables
 $x_1, \dots, x_n$

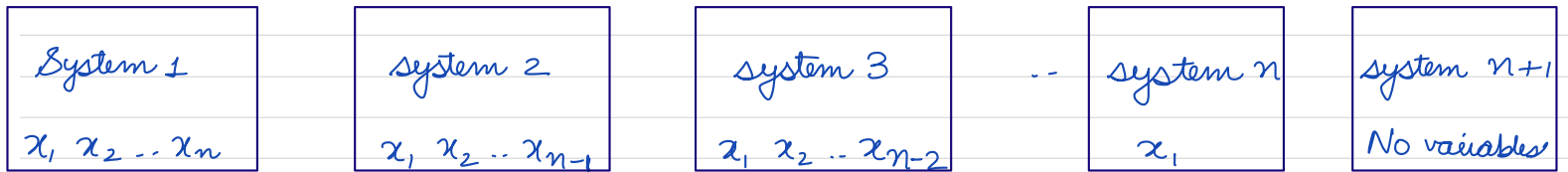
system 1

$O(m^2)$ inequalities
 $(n-1)$ variables
 $x_1, \dots, x_{n-1}$

system 2

system 1 is feasible $\iff$ system 2 is feasible.

Proof Follows from Claim 1 + Claim 2.



• System 1 is feasible $\iff$ system 2 is feasible $\iff \dots \iff$ system n is feasible $\iff$ system $(n+1)$ is feasible

- Inequalities in system $(i+1)$ is linear combination of inequalities in System i
- So, inequalities system $(n+1)$ is formed by combination of inequalities in System 1.

IF ANY INEQUALITY OF SYSTEM $(n+1)$ IS INVALID, THEN THERE EXISTS CORRESPONDING λ 's.

Fourier Motzkin Elimination is helpful in showing existence of $(\lambda_1, \dots, \lambda_m)$ for an infeasible LP, but its time complexity is $O(m^{2^n})$.

Q. Can we compute $\lambda = (\lambda_1, \dots, \lambda_m)$ in polynomial time?

Solⁿ: If $Ax \geq b$ is infeasible then by Farkas's lemma, there exists $\lambda = (\lambda_1, \dots, \lambda_m)$ satisfying $\lambda^T A = 0^n$, $\lambda^T b = 1$.

$$\Rightarrow \lambda^T [A \ b] = [0^n, 1] \quad \text{or} \quad [A \ b]^T \lambda = [0^n, 1]^T$$

Now solⁿ to $[A \ b]^T \lambda = [0^n, 1]^T$ can be obtained in polynomial time by Gaussian Elimination!

IMPLICATION: We can check in polynomial time whether a system $Ax \geq b$ is feasible or not by solving $[A \ b]^T \lambda = [0^n, 1]^T$

Feasibility Ques

$$Ax \geq b$$

Feasible /
Infeasible

(Yes/No)

If feasible
then find x

$$x \in \mathbb{Q}^n$$

Optimality Ques

$$\max c^T x$$

$$\text{s.t. } Ax \geq b$$

Find opt
value of $c^T x$

real number

Find feasible x
minimizing $c^T x$

$$x \in \mathbb{Q}^n$$

Can be solved by
Gaussian Elimination

We can
approximately compute value of opt
feasibility / infeasibility problem. (see next
pg)

Approximate Optimality using Feasibility

- Replace the objective function by a constraint $c^T x \geq t$, and search for feasible solution. (Assuming that we have reasonable estimates for upper/lower bounds on **objective fn**).
- By doing binary search on t , we can find a solution arbitrarily close to the true optimum.

Objective:
 $\max (x_1 + x_2)$

