COL 758: Advanced Algorithms

Lecture 20

Farka's Lemma (LP Duality)

A system of m inequalities of form

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

is infeasible **iff** $\exists\ \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ satisfying:

$$\lambda_1\Big(\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n\ \Big) \geq \lambda_1\Big(\ b_1\ \Big)$$
$$\vdots$$
$$\lambda_m\Big(\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\Big) \geq \lambda_m\Big(\ b_m\ \Big)$$

$$0 \qquad \geq \qquad 1$$

Proof of Feasibility:

An n-dim vector $(x_1, \ldots, x_n)$ satisfying all m inequalities.

Proof of Infeasibility

An m-dim vector $(\lambda_1, \ldots, \lambda_m)$ that on multiplying by m inequalities and summing gives $0 \geq 1$.

Farka's Lemma Proof by Fourier-Motzkin elimination (An Example)

Input system:

$$x - 2y + z \geq 3$$
$$2x - 5y + 4z \geq 10$$
$$-3x + 3y - 3z \geq 12$$
$$-2x + 14y - 4z \geq -4$$
$$-10y + z \geq -15$$

$$x - 2y + z \geq 3$$
$$x - 2.5y + 2z \geq 5$$
$$-x + y - z \geq 4$$
$$-x + 7y - 2z \geq -2$$
$$-10y + z \geq -15$$

$$x \geq 2y - z + 3$$
$$x \geq 2.5y - 2z + 5$$
$$y - z - 4 \geq x$$
$$7y - 2z + 2 \geq x$$
$$-10y + z \geq -15$$

$$\min(y - z - 4,\ 7y - 2z + 2) \geq \max(2y - z + 3,\ 2.5y - 2z + 5)$$
$$-10y + z \geq -15$$

$$y - z - 4 \geq 2y - z + 3$$
$$y - z - 4 \geq 2.5y - 2z + 5$$
$$7y - 2z + 2 \geq 2y - z + 3$$
$$7y - 2z + 2 \geq 2.5y - 2z + 5$$
$$-10y + z \geq -15$$

- The *input system* is feasible iff *new system* is feasible.
- The *new system* is obtained by linear combination of inequalities in *input system*.

Farka's Lemma Proof by Fourier-Motzkin elimination (An Example)

Input

$$
\begin{aligned}
x - 2y + z &\geq 3 \\
2x - 5y + 4z &\geq 10 \\
-3x + 3y - 3z &\geq 12 \\
-2x + 14y - 4z &\geq -4 \\
-10y + z &\geq -15
\end{aligned}
$$

x, y, z

$\Rightarrow$

$$
\begin{aligned}
&\boxed{-7 \geq y} \\
&2z/3 - 6 \geq y \\
&y \geq 0.2z + 0.2 \\
&\boxed{y \geq 2/3} \\
&0.1z + 1.5 \geq y
\end{aligned}
$$

y, z

$\Rightarrow$

$$
\begin{aligned}
-7 &\geq 0.2z + 0.2 \\
2z/3 - 6 &\geq 0.2z + 0.2 \\
0.1z + 1.5 &\geq 0.2z + 0.2 \\
\boxed{-7} &\boxed{\geq 2/3} \\
2z/3 - 6 &\geq 2/3 \\
0.1z + 1.5 &\geq 2/3
\end{aligned}
$$

z

Not feasible (why?)

- The new system is obtained by linear combination of inequalities in input system.
  This proves Farka's lemma.

- On eliminating one variable, new system may have $\Omega(m^2)$ inequalities.
  Thus, time complexity is $O\left(m^{2^n}\right)$.

Farka's lemma Proof.

m inequalities n variables $x_1, \ldots x_n$	$O(m^2)$ inequalities $(n-1)$ variables $x_1, \ldots x_{n-1}$
system 1	system 2

Conversion from system 1 to system 2:

- Write inequalities containing x_n as

$$\underbrace{F(x_1, \ldots x_{n-1})}_{\text{let these be } k \text{ in count}} \geq x_n \quad \text{or} \quad \underbrace{x_n \geq F(x_1, \ldots x_{n-1})}_{\text{let these be } q \text{ in count}}.$$

- Create kq new inequalities by eliminating x_n.

Claim 1

If system 1 is feasible $\Rightarrow$ system 2 is feasible

Proof:

Let $(x_1, \ldots x_n)$ be a sol^n to system 1. Then, $(x_1, \ldots x_{n-1})$ will be a sol^n to system 2. Because, inequalities in system 2 are obtained by combination of inequalities in system 1.

Claim 2

If system 2 is feasible $\Rightarrow$ system 1 is feasible

Proof:

Let $(x_1, \ldots x_{n-1})$ be a sol^n to system 2.

Recall: While eliminating x_n we obtained upper & lower bounds on x_n as

$$\min\{\_\ k \text{ inequalities}\ \_\} \geq x_n \geq \max\{\_\ q \text{ inequalities}\ \_\}$$

Bcoz the new system is satisfiable, we can find an appropriate value of x_n.

m inequalities
n variables
$x_1, \ldots x_n$

system 1

$O(m^2)$ inequalities
$(n-1)$ variables
$x_1, \ldots x_{n-1}$

system 2

Theorem

system 1 is feasible iff system 2 is feasible.

Proof Follows from Claim 1 + Claim 2.

System 1	system 2	system 3	..	system n	system $n+1$
$x_1, x_2 \ldots x_n$	$x_1, x_2 \ldots x_{n-1}$	$x_1, x_2 \ldots x_{n-2}$		x_1	No variables

- System 1 is feasible $\Leftrightarrow$ system 2 is feasible $\Leftrightarrow$ - - - $\Leftrightarrow$ System n is feasible $\Leftrightarrow$ system $(n+1)$ is feasible
- Inequalities in system $(i+1)$ is linear combination of inequalities in System i
- So, inequalities system $(n+1)$ is formed by combination of inequalities in System 1.

IF ANY INEQUALITY OF SYSTEM $(n+1)$ IS INVALID, THEN THERE EXISTS CORRESPONDING λ's.

Fourier Motzkin Elimination is helpful in showing existence of $(\lambda_1, \dots \lambda_m)$ for an infeasible LP, but its time complexity is $O(m^{2^n})$.

Q. Can we compute $\lambda = (\lambda_1, \dots \lambda_m)$ in polynomial time?

Sol^n: If $Ax \geq b$ is infeasible then by Farka's lemma, there exists $\lambda = (\lambda_1, \dots \lambda_m)$ satisfying $\lambda^T A = 0^n$, $\lambda^T b = 1$.

$$\Rightarrow \quad \lambda^T [A \; b] = [0^n, 1] \quad \text{or} \quad [A \; b]^T \lambda = [0^n, 1]^T$$

Now sol^n to $[A\,b]^T \lambda = [0^n, 1]^T$ can be obtained in polynomial time by Gaussian Elimination!

IMPLICATION: We can check in polynomial time whether a system $Ax \geq b$ is feasible or not by solving $[A\,b]^T \lambda = [0^n, 1]^T$

Feasibility Ques

$Ax \geq b$

- Feasible / Infeasible
  (Yes / No)
- If feasible then find x
  $x \in \mathbb{Q}^n$

Can be solved by Gaussian Elimination

Optimality Ques

$$\max c^T x$$
$$\text{s.t. } Ax \geq b$$

- Find opt value of $c^T x$
  real number
- Find feasible x maximizing $c^T x$
  $x \in \mathbb{Q}^n$

We can approximately compute value of opt feasibility / infeasibility problem.

(see next pg)

Approximate Optimality using Feasibility

- Replace the objective function by a constraint $c^T x \geq t$, and search for feasible solution. (Assuming that we have reasonable estimates for upper/lower bounds on objective fn).

- By doing binary search on t, we can find a solution arbitrarily close to the true optimum.

Objective:
max $(x_1 + x_2)$