COL 758: Advanced Algorithms

Lecture 21

Farka's Lemma (LP Duality)

A system of m inequalities of form

Short form: $Ax \geq b$

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1$$
$$\vdots$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m$$

is infeasible **iff** $\exists\ \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ satisfying:

$$\lambda_1\left(a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \right) \geq \lambda_1\left(b_1 \right)$$
$$\vdots$$
$$\vdots$$
$$\lambda_m\left(a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n\right) \geq \lambda_m\left(b_m \right)$$

$$0 \qquad \geq \qquad 1$$

$\lambda^T A = 0, \quad \lambda^T b = 1$

Alternatively,

$$\underbrace{\lambda^T A = 0, \quad \lambda^T b \,\cancel{\geq}\, 0}$$

Primal and Dual LP

Primal LP

$$\min \ c^T x$$
$$\text{s.t.} \quad Ax \geq b$$

Let x^* be opt sol^n and,

$$\gamma = \min \{ c^T x \mid Ax \geq b \}$$

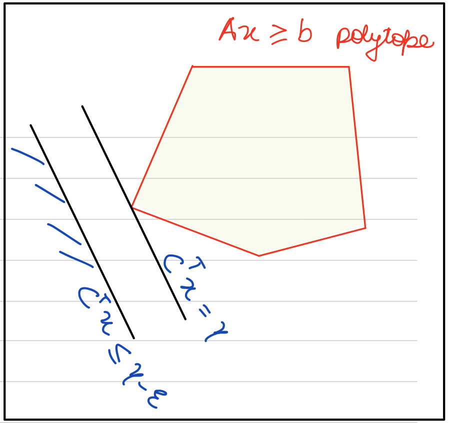
Dual LP

$$\max \ y^T b$$
$$\text{s.t.} \quad y^T A = c^T$$
$$y \geq 0$$

Dual-opt $\geq$ Primal-opt

$A_{m \times n}$

$$\text{Infeasible} \begin{cases} Ax \geq b \\ c^T x \leq \gamma - \varepsilon \end{cases}$$



$$\text{Infeasible} - \begin{bmatrix} A \\ -c^T \end{bmatrix} x \geq \begin{bmatrix} b \\ -\gamma+\varepsilon \end{bmatrix}$$

By Farkas Lemma,

$$[\underbrace{\lambda_1 \cdots \lambda_m}\ \lambda_0] \begin{bmatrix} A \\ -c^T \end{bmatrix} = 0 \qquad [\lambda_1 \cdots \lambda_m\ \lambda_0] \begin{bmatrix} b \\ -\gamma+\varepsilon \end{bmatrix} > 0$$

Let $\lambda = \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_m \end{pmatrix}$. Note $\lambda \geq 0^m$ and $\lambda_0 \not> 0$ (why?)

$$\Rightarrow \quad \lambda^T A - \lambda_0 c^T = 0 \quad \& \quad \lambda^T b > \lambda_0 (\gamma - \varepsilon)$$

$$\Rightarrow \boxed{\left(\frac{\lambda}{\lambda_0}\right)^T A = c^T, \quad \left(\frac{\lambda}{\lambda_0}\right)^T \geq 0, \quad \left(\frac{\lambda}{\lambda_0}\right)^T b > \gamma - \varepsilon, \ \forall \varepsilon}$$

Dual-opt $> \gamma - \varepsilon, \ \forall \varepsilon \Rightarrow$ Dual-opt $\geq \gamma$

Primal and Dual LP

Primal LP

$$\min\ c^T x$$
$$\text{s.t.}\quad Ax \geq b$$

Let x^* be opt sol^n and,

$y = \min \{c^T x \mid Ax \geq b\}$

Dual LP.

$$\max\ y^T b$$
$$\text{s.t.}\quad y^T A = c^T$$
$$y \geq 0$$

Claim 1: Dual-opt $\geq$ Primal-opt

Proof: last pg.

Claim 2: Primal-opt $\geq$ Dual-opt

Proof:

$$c^T x = y^T A x \geq y^T b \quad (\text{bcoz } y \geq 0)$$

$$\forall x, y. \qquad c^T x \geq y^T b$$

Let x^* = opt of primal, y^* = opt of dual,

Then

$$\text{Primal opt} = c^T x^* \geq {y^*}^T b = \text{dual-opt}$$

$\square$

We know,

$$\min\ c^T x \quad \text{s.t.}\ Ax \geq b \quad \xrightarrow{\text{dual}} \quad \max\ y^T b \quad \text{s.t.}\ y^T A = c^T,\ y \geq 0$$

H.W. 1

$$\min\ c^T x \quad \text{s.t.}\ Ax \geq b,\ x \geq 0 \quad \xrightarrow{\text{dual}} \quad \max\ y^T b \quad \text{s.t.}\ y^T A \leq c^T,\ y \geq 0$$

H.W. 2

$$\max\ y^T b \quad \text{s.t.}\ y^T A = c^T,\ y \geq 0 \quad \xrightarrow{\text{dual}} \quad \min\ c^T x \quad \text{s.t.}\ Ax \geq b$$

Transformation b/w different LPs

Short form:
$Ax \geq b$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1 \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m \end{aligned}$$

Standard Form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ &\vdots \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \\ x_1, \ldots x_n &\geq 0 \end{aligned}$$

Equational Form

Short form:
$Ax = b$
$x \geq 0$

How to transform Standard form to Equational form?

(H.W.)

- Introduce new variables
  - To make inequality → equality
  - x has no constraint $\quad x = f(y_1, y_2)$, $y_1, y_2 \geq 0$

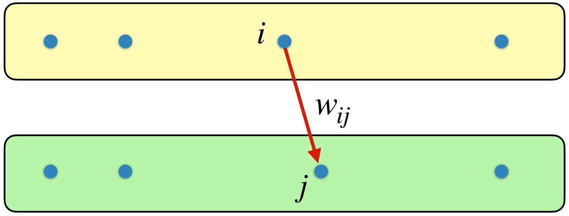
How to transform Equational form to Standard form?

$$\left.\begin{aligned} Ax &\geq b \\ -Ax &\geq -b \\ Ix &\geq 0^n \end{aligned}\right\} \Rightarrow \begin{bmatrix} A \\ -A \\ I \end{bmatrix} x \geq \begin{bmatrix} b \\ -b \\ 0^n \end{bmatrix}$$

Bipartite Perfect Matching

(People) $|U| = n$

(Jobs) $|J| = n$



w_{ij} = Cost person i is asking for doing job j.

Goal: Find perfect matching of size n with minimum weight.

Integer LP

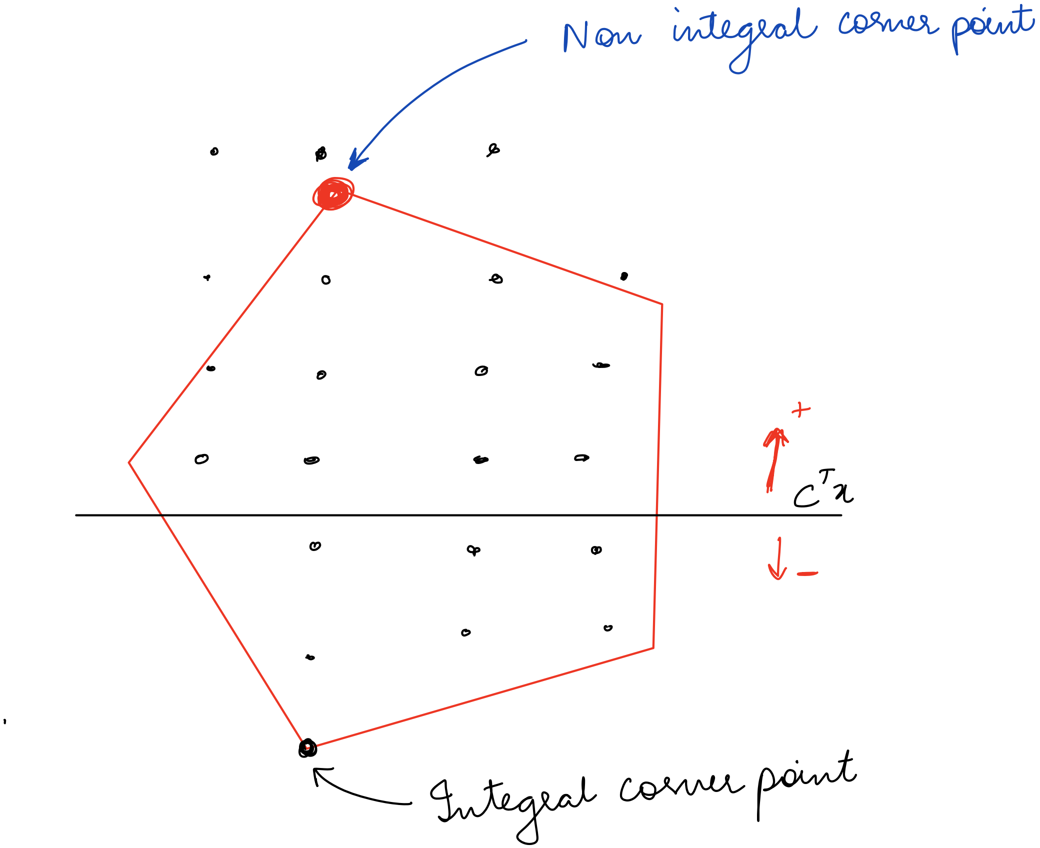
Variables
- $x_{ij} \in \{0, 1\}$

Constraints
- $\forall\, i \in U \quad \sum_{j \in OUT(i)} x_{ij} = 1$
- $\forall\, j \in J \quad \sum_{i \in IN(j)} x_{ij} = 1$

Relaxed LP

Variables
- $0 \leq x_{ij} \leq 1$

Constraints
- $\forall\, i \in U \quad \sum_{j \in OUT(i)} x_{ij} = 1$
- $\forall\, j \in J \quad \sum_{i \in IN(j)} x_{ij} = 1$



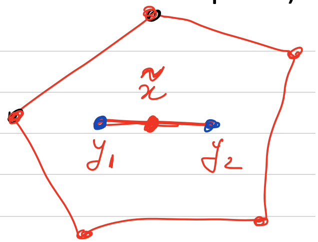
Integer LP (ILP)

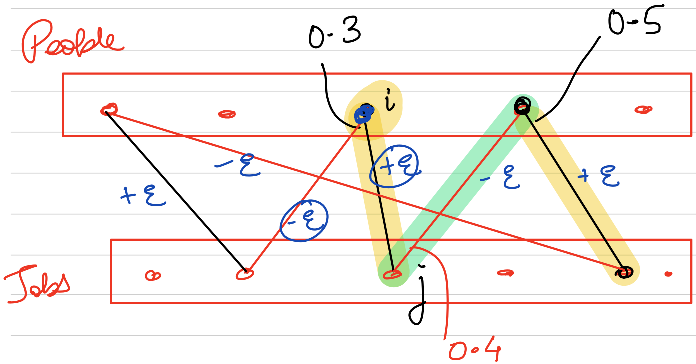
Bipartite Perfect Matching

Theorem: All extreme points of "relaxed LP" for Bipartite Perfect Matching are integral.

Proof: Let $\tilde{x}$ be a feasible solution, that is not integral. (Aim is to show that $\tilde{x}$ cannot be extreme point).

Take an edge (i,j) such that $\tilde{x}_{ij}$ is fractional.





compute a cycle C where all edges take fractional value.

$\exists \varepsilon$ s.t. value of edges of cycle $\in [\varepsilon, 1-\varepsilon]$

Def^n of y_1

Take $\tilde{x}$, add $+\varepsilon$ on odd edges of C, add $-\varepsilon$ on even edges of C

Def^n of y_2

Take $\tilde{x}$, add $+\varepsilon$ on even edges of C, add $-\varepsilon$ on odd edges of C

Claim 1: y_1, y_2 lie in feasible LP.

Claim 2: $\tilde{x} = \dfrac{y_1 + y_2}{2}$, so $x \neq$ vertex.