Applications of L.P.

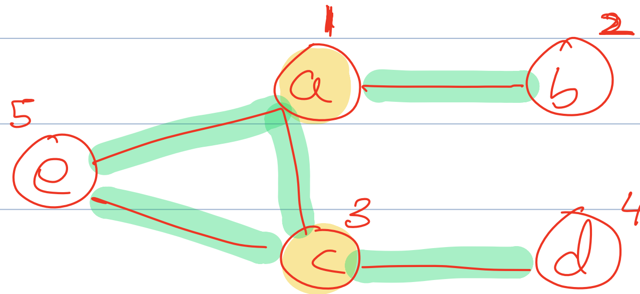
Vertex cover Problem

$G = (V, E)$ is given.

We have to find a set $S \subseteq V(G)$ of smallest size s.t. $|S \cap \{i,j\}| \geq 1 \quad \forall (i,j) \in E(G)$.

$(1, 0, 1, 0, 0)$

vector for vertex-cover $\{a, c\}$



vertex-cover $= \{a, c\}$

Formulated Integer Program

$$x_i = \begin{cases} 1 & \text{if vertex } v_i \text{ is chosen in vertex cover} \\ 0 & \text{o/w} \end{cases}$$

IP

$(x_1 \ldots x_n)$ — binary vector of size n.

$$\min \sum_{i=1}^{n} x_i$$

In general NP-hard.

$$x_i + x_j \geq 1, \quad \forall (i,j) \in E(G)$$

$$x_i \in \{0, 1\} \quad \forall \text{ vertices } i \text{ in } G$$

"Relaxed L.P." ← solvable in poly time

$$\boxed{\begin{aligned} &\min \sum_{i=1}^{n} x_i \\ &x_i + x_j \geq 1, \quad \forall (i,j) \in E(G) \\ &\underline{0 \leq x_i \leq 1} \quad \forall \text{ vertices } i \text{ in } G \end{aligned}}$$

"Lecture 2: Relaxed LP to find 2-approximation of vertex-cover."

$(x_1^* \; x_2^* \; \ldots \; x_n^*)$ — optimal of LP

new vector

$(\hat{x}_1 \; \hat{x}_2 \; \ldots \; \hat{x}_n)$

stretch is at most 2.

$$\hat{x}_i = \begin{cases} 1 & \text{if } x_i^* \geq 1/2 \\ 0 & \text{o/w} \end{cases}$$

Approx factor of 2

→ $\displaystyle\sum_{i=1}^{n} \hat{x}_i \leq 2 \sum_{i=1}^{n} x_i^* = 2\, LP_{opt} \leq 2\, IP_{opt}$

→ $S = \{ v_i \mid \hat{x}_i = 1 \}$ will be a vertex cover.

correctness claim

$$\boxed{\begin{gathered} \forall (i,j) \in E(G) \\ x_i^* \geq 1/2 \text{ or } x_j^* \geq 1/2 \end{gathered}}$$

"SET COVER PROBLEM"

Given : Set $\underbrace{S_1 \;\; S_2 \; .. \; S_n}_{\text{sets}}$ and $U = (S_1 \cup S_2 \cup .. \cup S_n)$

universe

Find : Smallest $I \subseteq [1, n]$ s.t. $\bigcup_{i \in I} S_i$ = Universe U

Integer Prog. → Relaxed LP → Use LP to find some approximation.

Eg. $S_1 = \{1, 2, 3\}$ $\quad S_2 = \{2, 3, 4\}$ $\quad S_3 = \{4, 5\}$ $\quad S_4 = \{5\}$

$U = [1, 5]$ is universe.

(S_1, S_3) – set-cover of smallest possible size

Eg.

wt = 2	wt = 3	wt = 100	wt = 4
$S_1 = \{1, 2, 3\}$	$S_2 = \{2, 3, 4\}$	$S_3 = \{4, 5\}$	$S_4 = \{5\}$

$U = [1, 5]$ is universe.

(S_1, S_2, S_4) will be a set cover of minimum weight.

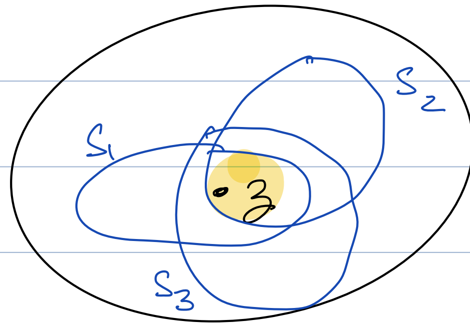
Integer Program

obj $\min \sum_{i=1}^{n} (w_i \cdot x_i)$

$x_i = \begin{cases} 1 & \text{if } S_i \text{ is chosen in set cover} \\ 0 & \text{o/w} \end{cases}$

(*) $\sum_{(i \mid z \in S_i)} x_i \geq 1 \quad \forall z \in U$

(*) $x_i \in \{0, 1\}$



$\boxed{w_i = \text{weight of set } S_i}$

$x_1 + x_2 + x_3 \geq 1$

Relaxed LP.

obj $\min \sum_{i=1}^{n} (w_i \cdot x_i)$

(*) $\sum_{(i \mid z \in S_i)} x_i \geq 1 \quad \forall z \in U$

(*) $0 \leq x_i \leq 1 \quad \forall i \in [1, n]$ $\quad n =$ no. of sets.

$(i, j) \quad x_i + x_j \geq 1$ ← vertex cover problem.

at least one of them $\geq 1/2$

Take element z. Suppose z lies in sets $S_1 \dots S_k$.

$x_1 + x_2 + \dots + x_k \geq 1$ we can say at least one $x_i \geq \frac{1}{k}$

Possible Rounding $\begin{cases} x_i \geq 1/k \longrightarrow 1 \\ x_i \not\leq 1/k \longrightarrow 0 \end{cases}$ (Round)

will give a stretch "k"

Algo

"Randomized Rounding"

1. Initialize $I = \emptyset$
2. Compute an opt sol^n $x^* = (x_1^*, \dots, x_k^*)$ to relaxed L.P.
3. For $i = 1$ to n:
   add i to I w.p. x_i^*
4. Return I

Ques. Consider $z \in U$. What is expected number of set covering the element z.

i.e. $E\left(\left|\{ i \mid z \in S_i \} \cap I\right|\right)$?

Ans: ≥ 1

Proof. Suppose $S_1 \ldots S_k$ contains z.

Def r.v. $R_i = \begin{cases} 1 & \text{if } S_i \text{ is included} \\ 0 & \text{o/w} \end{cases}$

$$E[R_i] = x_i^*$$

$$R = R_1 + \cdots + R_k \qquad E[R] = \overbrace{\sum_{i=1}^{k} x_i^*}^{\text{constant.}} \geq 1.$$

Sol^n not complete

Reason: Suppose z is covered by 2 sets S_1, S_2 and $x_1^* = x_2^* = 1/2$.

$$\text{Prob}(z \text{ is not covered}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

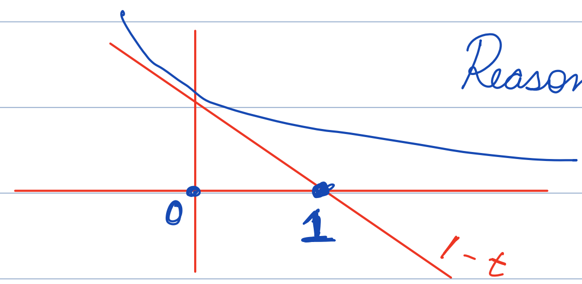
CLAIM: Consider an element $z \in U$.
In randomized roundg z is covered w.p. $\geq 1 - 1/e$

Proof: Let us suppose z lies in k sets, say $S_1 \ldots S_k$.

Then in LPopt sol^n x^* we have $x_1^* + \ldots + x_k^* \geq 1$

$$\text{Prob}(z \text{ is not covered}) = (1 - x_1^*) \ldots (1 - x_k^*)$$

$$1 - t \leq e^{-t}, \quad \forall\, t \in [0,1]$$



Reasoning 1

$$e^{-t} = 1 - t + \underbrace{\frac{t^2}{2!} - \frac{t^3}{3!} \cdots}_{\text{dominant term}}$$

Reasoning 2

$$\text{Prob}(z \text{ is not covered}) \leq e^{-x_1^*} e^{-x_2^*} \ldots e^{-x_k^*} \leq 1/e$$

$$\therefore \text{Prob}(z \text{ is covered}) \geq 1 - 1/e, \quad \forall z \in U. \quad \square$$

Ques

* How many times should step 3 be repeated, to get w.h.p.

* what will be the approx. factor ?