Set cover Problem

(unweighted)

$S_1, \ldots, S_n, \quad U$

$x_1 \quad\quad x_n$

Primal

$$\min \sum_{i=1}^{n} x_i \quad \text{s.t.}$$

$$\sum_{i \mid z \in S_i} x_i \geq 1 \quad \forall z \in U$$

$$x_i \geq 0$$

Dual

$$\max \sum_{z \in U} y_z \quad \text{s.t.}$$

$$\sum_{z \mid z \in S_i} y_z \leq 1 \quad \forall i \leq n$$

$$y_z \geq 0$$

Primal

$$\min c^T x$$
$$Ax \geq b$$
$$x \geq 0$$

Dual

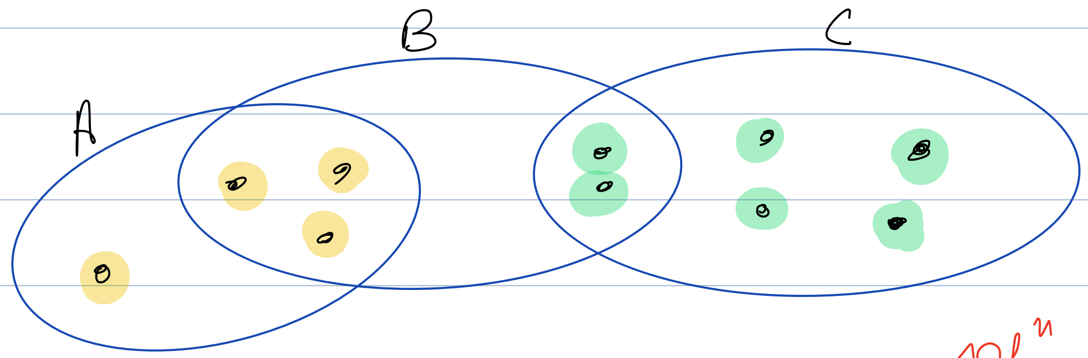
$$\max y^T b$$
$$A^T y \leq c$$
$$y \geq 0$$

Greedy Algo. (for unweighted case)

1. $I \leftarrow \phi$
2. While $\exists$ an uncovered element.
   - $i \leftarrow$ index of set S_i with largest no. of uncovered elements. ///
   - $I = I \cup \{i\}$

   (cost effec. of set picked)
3. Return I

Eg.

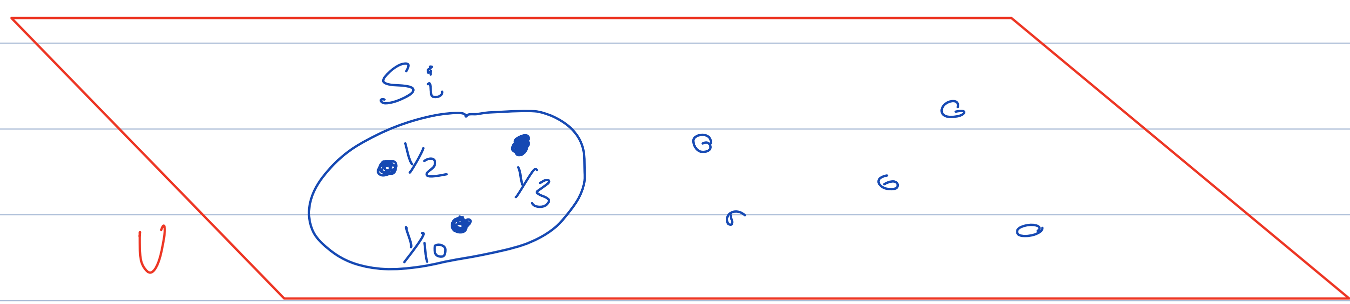


$sol^n = \{C, A\}$

Aim: To show "approx-factor" of Greedy Set Cover is $(\ln|U| + 1)$

Take set "S" in output of greedy set cover algo.

$\boxed{\text{cost-effectiveness}(S) := \text{No of uncovered elements in } U \text{ which were covered on adding } S \text{ in } sol^n}$



constraint in dual $\longrightarrow \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{10} \le 1\right)$

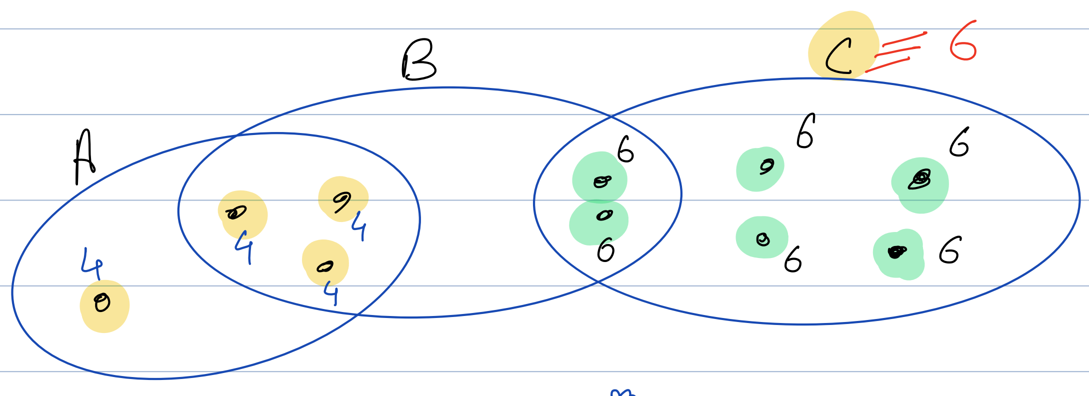
Let,

$(z_1, \dots, z_m)$ ← ordering in which elements are covered

c_j ← cost effectiveness of set that covered element z_j

H.W.: CLAIM: $\sum_{j=1}^{m} \frac{1}{c_j} = |I|$ output sol^n

Eg.



$\sum_{j=1}^{m} c_j = 6 \cdot \frac{1}{6} + 4 \cdot \frac{1}{4} = 2$

sol^n = $\{C, A\}$

H.W. 2 $\forall j = 1$ to m

$$y = (\underbrace{0 \ldots\ldots 0}_{j-1} \quad \underbrace{1/c_j \qquad\qquad 1/c_j}_{m-(j-1)}) \text{ is}$$

feasible to Dual LP.

Proof

Take any set S_i, we need to show

$$\sum_{z \in S_i} y_z \le 1$$

$$\iff \underbrace{|S_i \cap \{z_j \ldots z_m\}| \le c_j}$$

Can't be violated bcoz

if $|S_i \cap \{z_j \ldots z_m\}| \ge c_j + 1$

then to cover z_j we should have used set S_i — □

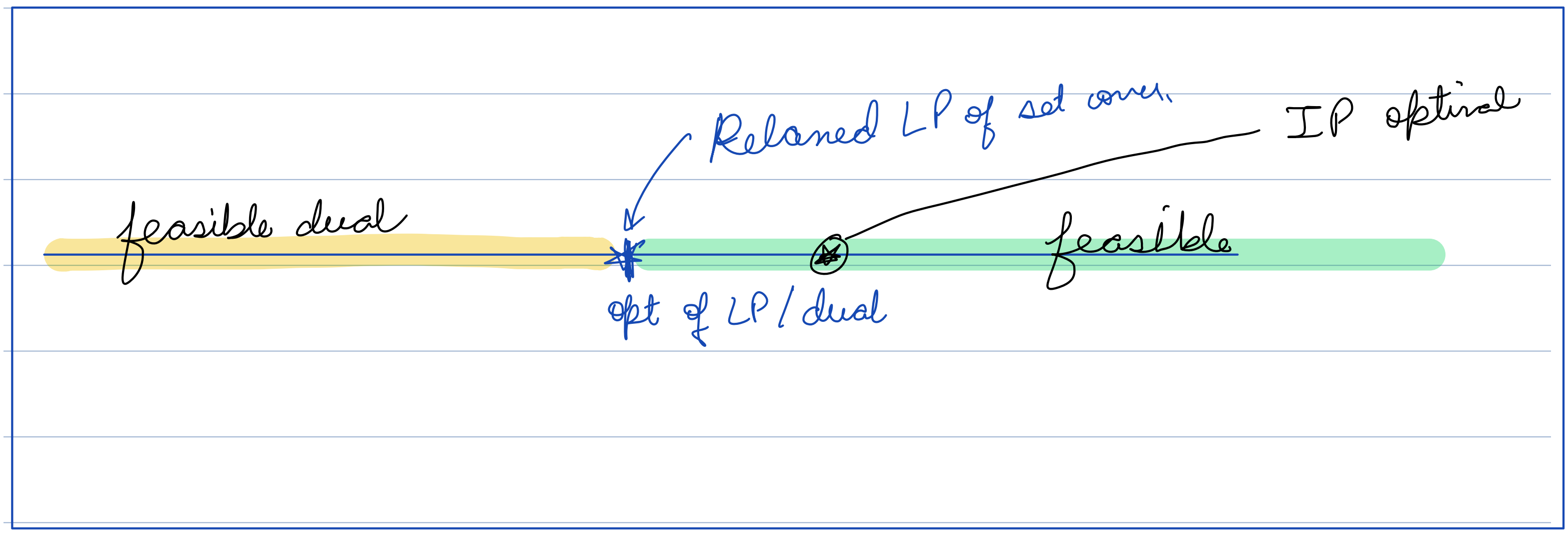
So, we have shown:

① $$\sum_{j=1}^{m} \frac{1}{c_j} = |\underbrace{I}_{\text{output set}}|$$

② $\forall\, j = 1$ to m

$$y = \Big(\underbrace{0 \,\cdots\, 0}_{j-1} \;\; \underbrace{1/c_j \qquad\qquad 1/c_j}_{m-(j-1)} \Big) \text{ is}$$

feasible to Dual LP.



$$|I| = \sum_{j=1}^{m} \frac{1}{c_j} = \sum_{j=1}^{m} \boxed{\frac{m-(j-1)}{c_j}} \cdot \frac{1}{m-(j-1)}$$

$$\leq \sum_{j=1}^{m} (\text{opt-LP}) \cdot \frac{1}{m-(j-1)}$$

$$\leq \boxed{\sum_{j=1}^{m} \frac{1}{m-(j-1)}} * \left(\text{opt-set-cover size}\right)$$

$m = |U|$

$$\leq \underbrace{\ln(|U|)+1}_{} * \text{opt set cover size}$$

approx factor of Greedy Set Cover..

Max Flow Problem

Primal of (s,t) - max - flow

$\downarrow$

Dual

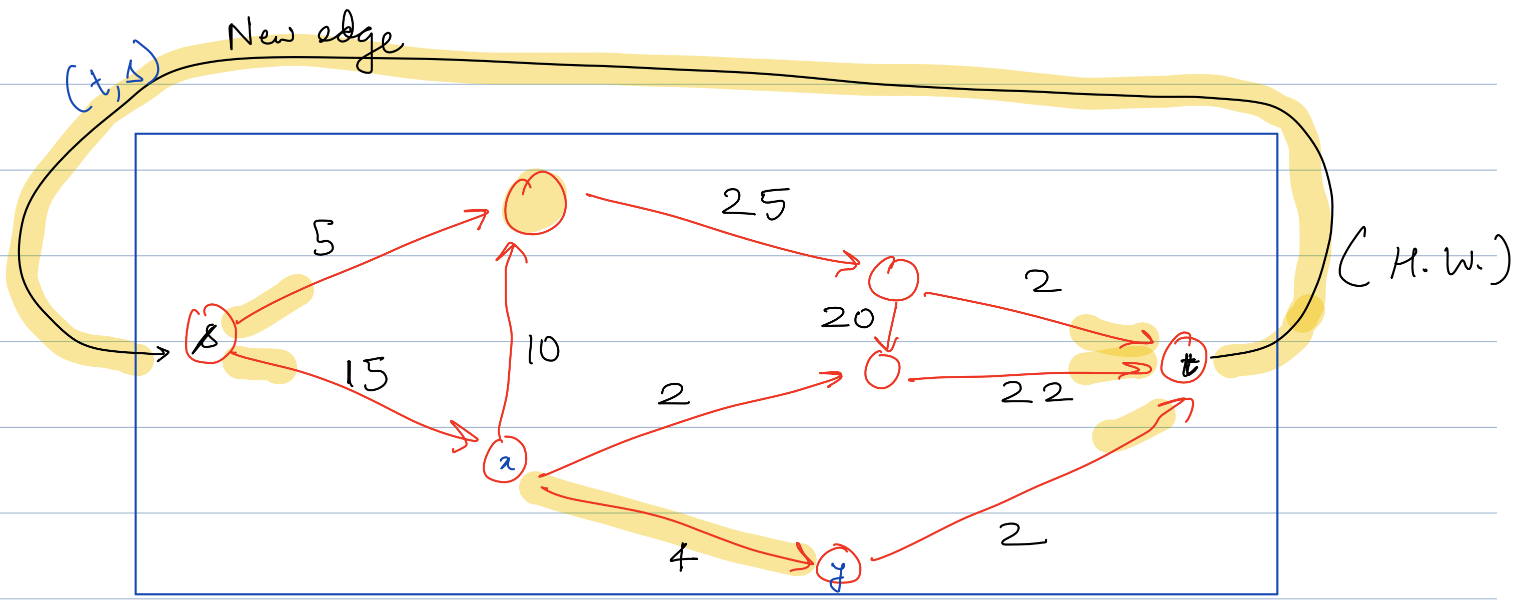
$\downarrow$

Dual $\equiv$ (s,t) - min - cut.

Max (s,t) - flow.

<u>Given</u>: A directed $G=(V, E, c: E \to \mathbb{R}^{+})$, $s, t \in$

Find: max s-t flow

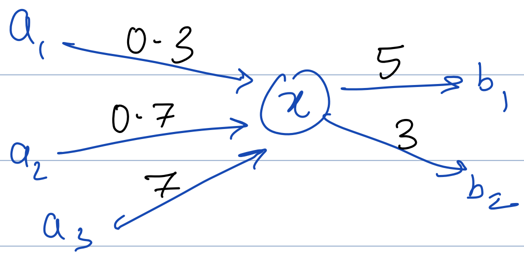


L.P (1)

$$\max\left(\sum_{b \in out(s)} f_{s,b}\right)$$

s.t. → $0 \leq f_{x,y} \leq c_{x,y} \quad \forall (x,y) \in E$

$$\sum_{a \in in(x)} f_{a,x} = \sum_{b \in out(x)} f_{x,b} \quad \forall x \neq s,t.$$



Max-flow can be solved in poly-time using LP solvers.

$$\begin{array}{c} \max\ (b^T y) \\ A y \leq c \\ y \geq 0 \end{array} \longleftrightarrow \begin{array}{c} \min\ (c^T x) \\ A^T x \geq b \\ x \geq 0 \end{array}$$

L.P (2)

$$\max\left(f_{t,s}\right)$$

s.t. → $0 \leq f_{x,y} \leq c_{x,y} \quad \forall (x,y) \in E \cup \cancel{\{(t,s)\}}$

Done

$$\sum_{a \in in(x)} f_{a,x} - \sum_{b \in out(x)} f_{x,b} = 0 \quad \forall x$$

L.P (3)

$$\max \left(f_{t,s} \right)$$

s.t. → $0 \leq f_{x,y} \leq c_{x,y} \quad \forall (x,y) \in E \cancel{\cup \{t,s\}}$

→ $$\sum_{a \in in(x)} f_{a,x} - \sum_{b \in out(x)} f_{x,b} \leq 0 \quad \forall x$$

CLAIM – opt of LP(1), LP(2), LP(3) are identical.

Proof – (H.W.)