Max (s,t) - flow

LP (2)

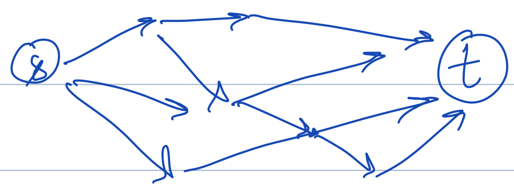
$$\max \left(f_{t,s} \right)$$

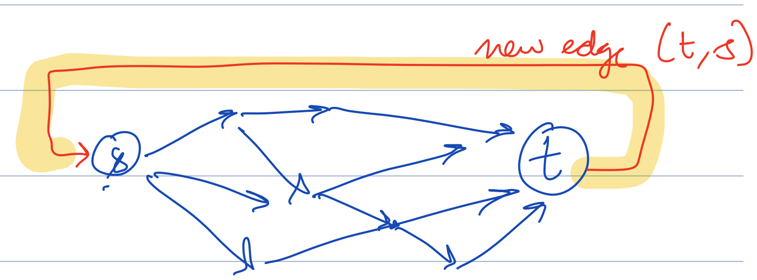
s.t. $\quad f_{a,b} \le c_{a,b} \qquad \forall (a,b) \in E(G)$

$$\forall v, \quad \underbrace{\sum_{a \in in(v)} f_{a,v}}_{in\text{-}flow(v)} - \underbrace{\sum_{b \in out(v)} f_{v,b}}_{out\text{-}flow(v)} \le 0 \qquad \forall v \in V(G)$$

$$f_{a,b} \ge 0 \qquad \forall (a,b) \in E(G)$$

$G \longrightarrow G_{new}$





Ques What is dual of linear program above ?

$$\max \left(f_{t,s} \right)$$

$$\text{s.t.} \quad f_{a,b} \leq c_{a,b} \qquad \forall \, (a,b) \in E(G)$$

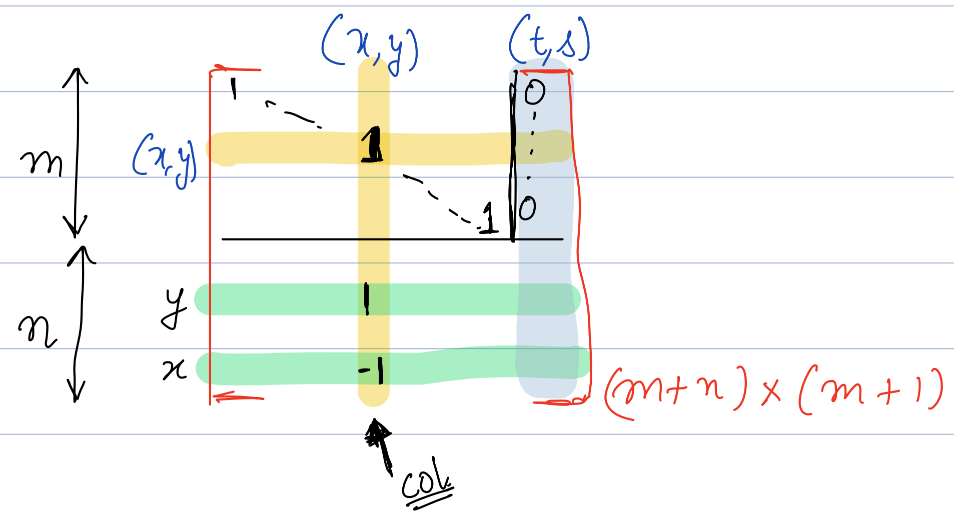
$$\forall v, \quad \underbrace{\sum_{a \in in(v)} f_{a,v}}_{in\text{-}flow(v)} - \underbrace{\sum_{b \in out(v)} f_{v,b}}_{out\text{-}flow(v)} \leq 0 \qquad \forall \, v \in V(G)$$

$$f_{a,b} \geq 0 \qquad \forall \, (a,b) \in E(G)$$

$\min \; c^T x$		$\max \; y^T b$
s.t. $\; A^T x \leq b$	$\longleftrightarrow$	s.t. $\; A y \leq c$
$x \geq 0$		$y \geq 0$

* Total no. of inequalities $= m + n = |V(G)| + |E(G)|$
* Total no of variables $= m + 1 = |E(G_{new})|$

A



# of variables in Dual $= m + n$ — $d_{x,y}$ ← edge variables; p_x ← vertex variables

# of inequalities in Dual $= m+1$

Dual

$$\min \left(\sum_{(x,y)\in E(G)} c_{x,y}\, d_{x,y} \right)$$

s.t.

$m+1$ inequalities

$$d_{x,y} + p_y - p_x \geq 0$$
$$p_s - p_t \geq 1$$

$$d_{x,y} \geq 0 \qquad \forall (x,y) \in E(G)$$
$$p_x \geq 0 \qquad \forall x \in V(G)$$

$$\begin{array}{cl} \begin{bmatrix} d \\ p \end{bmatrix} & \begin{array}{l} \updownarrow m \\ \updownarrow n \end{array} \end{array}$$

↑ variable vector

We will show this is relaxed LP of (s,t) - min - cut problem.

(s,t) - Cut Problem

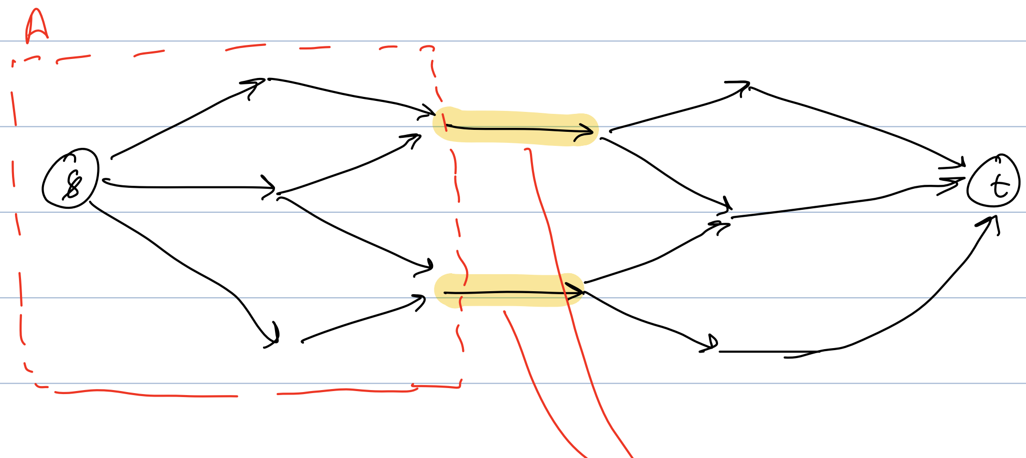
Given: A directed $G = (V, E, c : E \to \mathbb{R}^+)$, $s, t \in V$.

Aim: To disconnect source s and sink t.

Defn 1

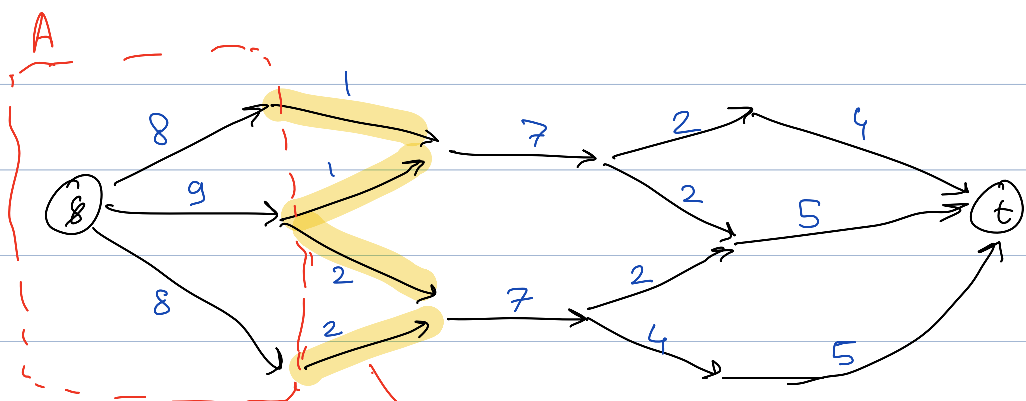
(s,t) - cut : Set of edges E' s.t. there is no s to t path in $G \setminus E'$.

Eg.



(s,t) - cut of least size

Eg.



(s,t) - cut of min - weight

Wt of (s,t) - cut $= 1 + 1 + 2 + 2.$

Def^n 2

(s,t)-cut : A partition (A, A^c) of $V(G)$.
s.t.

$s \in A$ and $t \in A^c$.

size of cut (A, A^c) = No of edges going from A to A^c.

$$\text{wt of cut } (A, A^c) = \sum_{\substack{(x,y) \in E(G) \\ x \in A \\ y \in A^c}} wt(x,y)$$

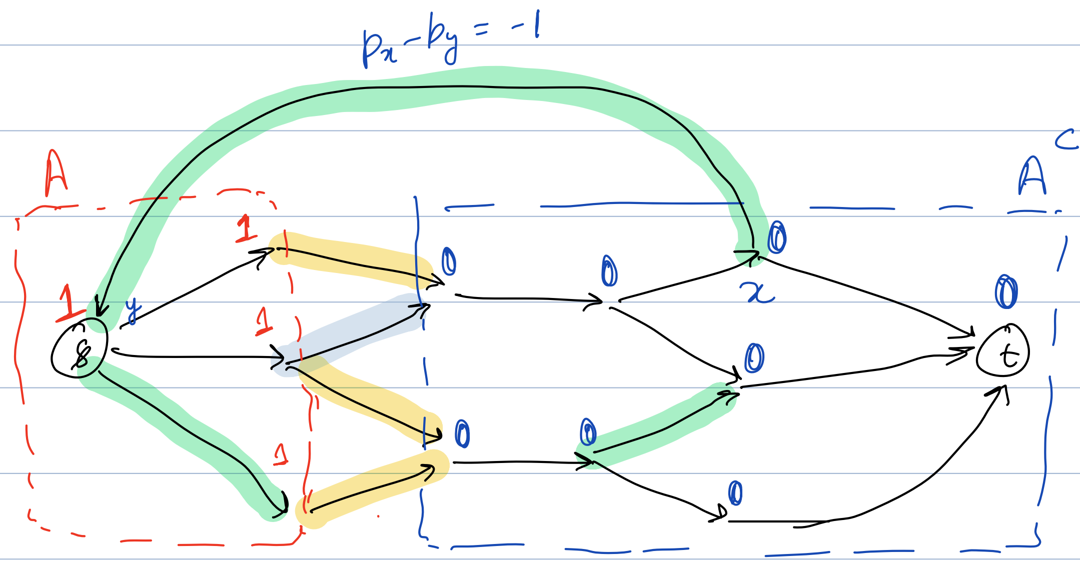
CLAIM An (s,t)-cut E' $\equiv$ A partition of $V(G)$, say (A, A^c) s.t. $s \in A$, $t \in A^c$

Proof: If (A, A^c) is given then E' = edges going from A to A^c

If E' is given, then set A = vertices reachable from s in $G \setminus A$

A^c = vertices not reachable from s in $G \setminus A$ $\square$

Eg.



mathematical form of min cut.

$$\boxed{d_{x,y} = \max\{0, p_x - p_y\}}$$

$\geq$

Obj : $\min\left(\sum_{(x,y) \in E(G)} c_{x,y} \cdot d_{x,y} \right)$

$$\left\{ \begin{array}{ll} p_x \in \{0, 1\} & \forall\, x \in V(G) \\ p_s = 1 & \\ p_t = 0 & \end{array} \right.$$

$d_{x,y} \geq 0 \qquad \forall\, (x,y) \in E(G)$

$d_{x,y} \geq (p_x - p_y)$

$$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}_{\text{Integer Program for } (s,t)-\text{min-cut.}}$$

Relaxed L.P.

Relaxed LP for (s,t)-min cut

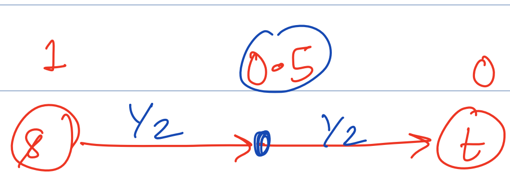
$$\min \left(\sum_{(x,y) \in E(G)} c_{x,y} \cdot d_{x,y} \right)$$

$$\begin{cases} p_x \geq 0, \quad \forall\, x \in V(G) \\ p_s = 1 \\ p_t = 0 \end{cases}$$

$$\begin{cases} d_{x,y} \geq 0 \\ d_{x,y} \geq (p_x - p_y) \end{cases} \quad \forall\, (x,y) \in E(G)$$

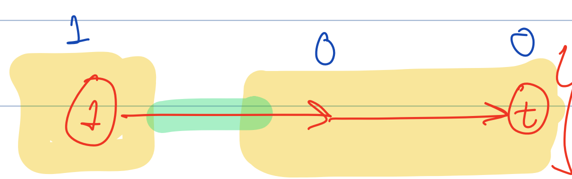
Dual of (s,t)-max flow.

E.g.

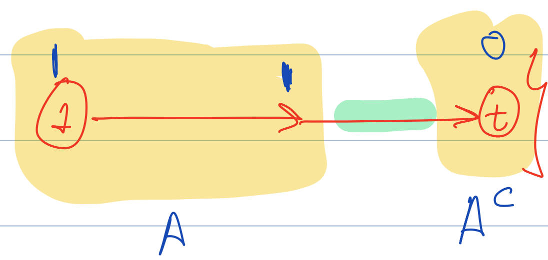


optimal of relaxed L.P.

Related



I P



I P

Ques (1) How to use relaxed LP of (s,t)-min cut problem to get a sol^n of IP for (s,t)-min-cut

Ques (2) opt relaxed LP $\equiv$ opt IP for (s,t)-min cut.

Q3 Use Q2 to prove Max Flow Min Cut Thm.