Zero Sum Game

Two players A — m strategies $[1, .. m]$

B — n strategies $[1, -- n]$

$\forall\ i \in [1, m],\ j \in [1, n],$

$$\text{Pay-off}(A \text{ for } i,j) + \text{Pay-off}(B \text{ for } i,j) = 0$$

To define a game we need

$P \leftarrow$ pay off of A

$(-P) \leftarrow$ pay off of B

Randomized (Mixed stategy of A

$A \leftarrow (x_1, -- x_m)$ x_i = prob of choosing strategy 'i'

$$\sum_{i=1}^{m} x_i = 1$$

$$x_i \geq 0$$

Eg. Rock-Paper-Scissor game

$R_o > S_c$

$S_c > P_a$

$P_a > R_o$

A \ B	R_o	P_a	S_c
R_o	0 ✓	−1 ✓	1 ✓
P_a	1	0	−1
S_c	−1	1	0

What A wins

A \ B	R_o	P_a	S_c
R_o	0	1	−1
P_a	−1	0	1
S_c	1	−1	0

What B wins

H.W.

$x = (1/3, 1/3, 1/3) \quad y = (1/3, 1/3, 1/3)$ is a NE.

CLAIM: If B has fixed its strategy to be $(1/3, 1/3, 1/3)$ then for all strategies of A

i.e. $\forall\, x = (x_1, x_2, x_3) \geq 0$ with $x_1 + x_2 + x_3 = 1$

Expected Pay off $(A) = 0$

If A plays $x = (x_1, \dots, x_m)$ and

B plays $y = (y_1, \dots, y_n)$

$$\text{Exp Payoff of A} = \sum_{i,j} x_i\, y_j\, p_{ij} = \boxed{x^T P y}$$

$$\text{Exp Payoff of B} = \sum_{i,j} x_i\, y_j\, (-p_{ij}) = x^T (-P) y$$

Suppose B has declared its strategy to be $y = (y_1 \ldots y_n)$ then what is best strategy for A?

$$\alpha(y) = \max \left(x^T P y \right)$$

$$\text{s.t.} \quad \sum_{i=1}^{m} x_i = 1$$

$$x_1 \ldots x_m \geq 0$$

Suppose A has declared its strategy to be $x = (x_1 \ldots x_n)$ then what is best strategy for B?

$$\boxed{\begin{aligned} \beta(x) &= \min \left(x^T P y \right) \\ \text{s.t.} \quad &\sum_{j=1}^{n} y_j = 1 \\ &y_1 \ldots y_n \geq 0 \end{aligned}}$$

Best strategy for B if we know $x = (x_1 \ldots x_m)$

$$A = \underbrace{\begin{bmatrix} 1 \ldots & \ldots 1 \end{bmatrix}}_{n} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = 1$$

Dual variable $= x_0$

Dual LP

$$\boxed{\begin{aligned} &\max (x_0) \\ &\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1} x_0 \leq P^T x \end{aligned}}$$

$$\boxed{\begin{aligned} &\min c^T x \\ &Ax = b \\ &x \geq 0 \end{aligned}} \xrightarrow{\text{Dual}} \begin{aligned} &\max b^T y \\ &A^T y \leq c \\ &\cancel{y \geq 0} \end{aligned}$$

LP(A)

$$\max \; (x_0)$$

s.t.

$$\begin{bmatrix} P^T & - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{n\times 1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \\ \hline x_0 \end{bmatrix} \geq 0$$

$$\sum_{i=1}^{n} x_i = 1$$

$$x_1 \; \ldots \; x_n \geq 0$$

Best strategy for A.

$$\text{Best strategy for } A = \arg\max_x \Big(\underbrace{\min_y \left(x^T P y \right)}_{\text{gain of A if } x \text{ is decided}} \Big)$$

$$\text{Best strategy for } B = \arg\min_y \Big(\max_x \left(x^T P y \right) \Big)$$

Pay off of A

$$\min \left(x^T P y\right)$$

$$\text{s.t.} \quad \sum_{j=1}^{n} y_j = 1$$

$$y_1 \ldots y_n \geq 0$$

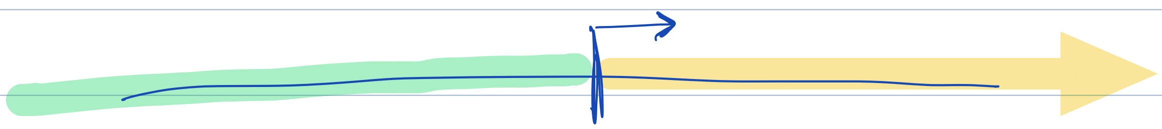
Best strategy for B if we know x

Pay off of A

$$\max \left(x_0\right)$$

$$\begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1} x_0 \leq P^T x$$

linear in $x_1 \ldots x_m$ & x_0



x_0

LP(B)

LP

$$\min \left(y_0\right)$$

$$\text{s.t.} \quad \left[P - \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{m \times 1} \right] \begin{bmatrix} y_1 \\ \vdots \\ y_m \\ y_0 \end{bmatrix} \leq 0$$

$$\sum_{j=1}^{n} y_j = 1$$

$$y_1 \cdots y_n \geq 0$$

Best strategy for B

⊗ If A has declared its strategy to be $(x_1, \cdots x_m)$

then best choice for B. is

$$\max_{x} \quad \min_{y} \left(x^T P y \right) \quad \text{s.t.} \quad y_1, \cdots y_n \geq 0, \quad \sum_{j=1}^{n} y_j = 1 \qquad \text{— LP 1}$$

$$\max_{x} \quad \max_{x_0} (x_0) \quad \text{s.t.} \quad \begin{bmatrix} x_0 \\ \vdots \\ x_0 \end{bmatrix}_{n \times 1} \leq P^T \underline{x} \qquad \text{— LP 2}$$

Ques If A has to declare its strategy then why should A choose that x for which

LP 2 has best sol^n?

Payoff of A = $x^T P y$.

LP(A) + LP(B) are dual of each other.