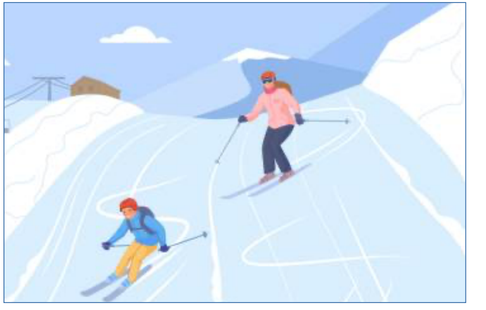
COL 758: Advanced Algorithms

Lecture 30

Ski-Rental

Ski Equipment:

- Rent per day = 100 INR
- Buy = 1,000 INR



k = Number of days you will Ski

Question: Should you buy / rent the Ski Equipment?

Offline Ski-Rental

Ski Equipment:

- Rent per day = 100 INR
- Buy = 1,000 INR

k = Number of days you will Ski

(Known in beginning)

Question: Should you buy / rent the Ski Equipment?

Solution: If ($k \leqslant 10$) then Rent, else Buy.

Cost: $\min(100k, 1000)$

“Online” Ski-Rental

Ski Equipment:

- Rent per day = 100 INR
- Buy = 1,000 INR

k = Number of days you will Ski $\left(\begin{array}{c}\text{Unknown, depends} \\ \text{on weather conditions}\end{array}\right)$

Question: Till when should you Rent the Ski Equipment (before buying)?

Design an algorithm for which $\alpha = \max_{k \geq 1} \left(\frac{\text{cost you pay}}{\text{opt cost if '}k\text{' was known}} \right)$ is minimised

"Online" Ski-Rental

Ski Equipment:

- Rent per day = 100 INR
- Buy = 1,000 INR

k = Number of days you will Ski (unknown)

Strategy 1: Buy on 8^{th} day

$\alpha = \frac{1700}{800}$

	$k \leq 7$	$k = 8$	$k = 9$	$k \geq 10$
Online	$100k$	1700	1700	1700
Offline	$100k$	800	900	1000
Ratio	1	$\frac{1700}{800}$	$\frac{1700}{900}$	$\frac{1700}{1000}$

Strategy 2: Buy on 10^{th} day

$\alpha = 1.9$

	$k \leq 9$	$k \geq 10$
Online	$100k$	1900
Offline	$100k$	1000

"Online" Ski-Rental

$\boxed{\text{Assumption: } B \text{ is integer}}$

Homework: What if B is not an integer ?

General Set-up:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski (unknown)

Offline Set-up (k is known)	If ($k \leqslant$ B) then Rent, else Buy. Cost: min(k, B)

Strategy for Online Setup: Buy on Bth day

	$k \lneq B$	$k \geq B$
Offline	k	B
Online	k	$(B-1) + B$

$$\alpha = \frac{2B-1}{B} = 2 - \frac{1}{B}$$

Is this the best deterministic strategy

"Online" Ski-Rental

General Set-up:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski (unknown)

Randomized algorithm

On the i^{th} day toss a coin (Prob(H) = p_i)

Rent with probability p_i

Buy with probability $(1 - p_i)$

$p_1 > p_2 > \dots > p_i > p_{i+1}$

Probability of Renting should decrease with time

Integer Program: Ski-Rental (Offline version)

General Set-up:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski (Known)

Integer Program

Renting Cost

Buying cost

Optimize $\quad \min \left(\underbrace{\sum_{j=1}^{k} Z_j}_{\text{Renting Cost}} + \underbrace{x \cdot B}_{\text{Buying cost}} \right)$

Subject to: $\quad Z_j \in \{0,1\}, \ \text{for } j \in [1,k]$

$x \in \{0,1\}$

$\boxed{Z_j + x \geqslant 1}, \ \text{for } j \in [1,k]$

Rent on day j, or
should have already bought

$$Z_j = \begin{cases} 1 & \text{rent on } j^{th} \text{ day} \\ 0 & \text{otherwise.} \end{cases}$$

$$x = \begin{cases} 1 & \text{buy some day} \\ 0 & \text{otherwise.} \end{cases}$$

LP relaxation: Ski-Rental (Offline version)

Primal

$$\begin{aligned} &\min\ c^T x \\ &Ax \geq b \\ &x \geq 0 \end{aligned}$$

Dual

$$\begin{aligned} &\max\ y^T b \\ &A^T y \leq c \\ &y \geq 0 \end{aligned}$$

General Set-up:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski

Primal Vector $= (z_1 \quad z_k \quad x)^T$

$$c^T = (1 \; \cdots \cdots \; 1 \quad B)$$

$$A = \begin{array}{c} y_1 \rightarrow \\ \\ \\ \\ y_k \rightarrow \end{array} \begin{bmatrix} 1 & & & & 1 \\ & 1 & & & 1 \\ & & 1 & & 1 \\ & & & 1 & 1 \\ & & & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Linear Program (P)

Optimize $\quad \min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $\quad Z_j \geqslant 0, \ \text{for } j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1, \ \text{for } j \in [1,k]$

(k) Constraints

(k+1) variables

Dual

variables $\quad y_1 \ y_2 \ \dots \ y_k \geq 0$

$$\max \sum_{j=1}^{k} (y_j)$$

(k+1) Constraints: $\quad y_1 \dots y_k \leq 1$

$$(y_1 + y_2 + \dots + y_k) \leq B$$

LP relaxation: Ski-Rental (Offline version)

General Set-up:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski

Dual optimal

$$y = (\underbrace{1 - - - - - 1}_{\min(k,B)}\ \underbrace{0 - - 0}_{k - \min(k,B)})$$

cost = min (k, B)

Linear Program (P)

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0,\ \text{for } j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1,\ \text{for } j \in [1,k]$

Dual Program (D)

Optimize $\max\left(\sum_{j=1}^{k} y_j\right)$

Subject to: $y_j \leqslant 1,\ \text{for } j \in [1,k]$

$y_1 + \cdots + y_k \leqslant B$

$y_j \geqslant 0,\ \text{for } j \in [1,k]$

Primal Feasible:

Case 1 – ($k \leq B$) : ($z_1 \dots z_k = 1$, $x = 0$)

Case 2 – ($k \geq B$) : ($z_1 \dots z_k = 0$, $x = 1$)

} cost = min (k, B)

← This also optimal (why?)

Primal and Dual in Online Setting

Linear Program (P)

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0,\ \text{for } j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1,\ \text{for } j \in [1,k]$

j^{th} Day:

Add new constraint: $Z_j + x \geqslant 1$

Ensure that: $Z_j \geqslant 0$

Variables we can modify on j^{th} Day: Z_j, x

H.W. Ques: Argue that we can only increment x

Dual Program (D)

Optimize $\max\left(\sum_{j=1}^{k} y_j\right)$

Subject to: $y_j \leqslant 1,\ \text{for } j \in [1,k]$

$y_1 + \cdots + y_k \leqslant B$

$y_j \geqslant 0,\ \text{for } j \in [1,k]$

j^{th} Day:

Add new variable: y_j and the bound $0 \leqslant y_j \leqslant 1$

Ensure that: $y_1 + \cdots + y_j \leqslant B$

Variables we can modify on j^{th} Day: y_j

Primal and Dual in Online Setting

Linear Program (P)

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0$, for $j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1$, for $j \in [1,k]$

Dual Program (D)

Optimize $\max\left(\sum_{j=1}^{k} y_j\right)$

Subject to: $y_j \leqslant 1$, for $j \in [1,k]$

$y_1 + \cdots + y_k \leqslant B$

$y_j \geqslant 0$, for $j \in [1,k]$

Remark: We cannot solve primal/dual optimally because constraints/variables are added in dynamic fashion!

Invariant (for each day): Dual-solution (D) $\leqslant$ Primal-solution (P) $\leqslant$ $\underline{\underline{\alpha}} \cdot$ Dual-solution (D)

<u>H.W. Ques:</u> Argue that our primal sol^n is α-competitive