COL 758: Advanced Algorithms

Lecture 31

"Online" Ski-Rental

Ski Equipment:

- Rent per day = 1
- Buy = B

k = Number of days you will Ski $\left(\begin{array}{c}\text{Unknown, depends}\\ \text{on weather conditions}\end{array}\right)$

Design an algorithm for which $\alpha = \max_{k\geq 1}\left(\frac{\text{cost paid by online algorithm}}{\text{opt cost if '}k\text{' was known}}\right)$ is minimised

Deterministic Strategy: Buy on Bth day

$\left(2-\frac{1}{B}\right)$ - competitive

H.W.

Det

Best $\alpha = 2-\frac{1}{B}$

Randomized Strategy: ??

Rand

$\alpha \approx 1.58$

Primal and Dual (OFFLINE)

Primal Program

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0,\ \text{for } j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1,\ \text{for } j \in [1,k]$

Dual Program

Optimize $\max\left(\sum_{j=1}^{k} y_j\right) \rightarrow \max\left(\sum_{y \in Y} y\right)$

Subject to: $y_j \leqslant 1,\ \text{for } j \in [1,k] \rightarrow y \leq 1, \forall y \in Y$

$y_1 + \cdots + y_k \leqslant B \rightarrow \sum_{y \in Y} y \leq B$

$y_j \geqslant 0,\ \text{for } j \in [1,k] \rightarrow y \geq 0, \forall y \in Y$

$Z_j = \begin{cases} 1 & \text{rent on Day } j \\ 0 \end{cases}$

$x = \begin{cases} 1 & \text{If we buy.} \\ 0 \end{cases}$

$Y = \{ y_1 \cdots y_k \}$

Primal Program

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0,\ \text{for } j \in [1,k]$

$x \geqslant 0$

$Z_j + x \geqslant 1,\ \text{for } j \in [1,k]$

Dual Program

Optimize $\max\left(\sum_{y \in Y} y\right)$

Subject to: $y \in [0,1],\ \text{for } y \in Y$

$\sum_{y \in Y} y \leqslant B$

Set $Y = \{y_j \mid j \in [1,k]\}$ ← $Y = \{y_1 \ldots y_k\}$

ONLINE

Day j: Add new constraint: $Z_j + x \geqslant 1$

Day j: Add new variable: y_j to set Y

Day 0 $\min\left(\sum_{j=1}^{k} Z_j + x.B\right)$

s.t. $Z_j \geq 0$

$x \geq 0$

Day 0 $\max\left(\sum_{y \in Y} y\right)$

$0 \leq y \leq 1 \quad \forall\, y \in Y$

$\sum_{y \in Y} y \leq B \quad (Y = \emptyset)$

Day 1 $Z_1 + x \geq 1$

Day 2 $Z_2 + x \geq 1$ ←

Day 1 Add y_1 to set Y

Day 2 Add y_2 to set Y

Primal Program

Day 0

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0$, for $j \in [1,k]$

$x \geqslant 0$

Day j: Add new constraint: $Z_j + x \geqslant 1$

Day 1 $\quad Z_1 = 1 \quad x = 0$

⋮

Day $(j \le B)$ $\quad Z_1 \ldots Z_j = 1 \quad x = 0$

CHANGE

Day $B+1$ $\quad Z_1 \ldots Z_{B+1} = 0 \quad x = 1$

In online setting, we can't maintain optimal sol^n to PRIMAL

Dual Program (Evolving with time)

Optimize $\max\left(\sum_{y \in Y} y\right)$

Subject to: $y \in [0,1]$, for $y \in Y$

$\sum_{y \in Y} y \leqslant B$

Set $Y = \varnothing$ "" $\quad Y = \{y_1\}$

Day j: Add new variable: y_j to set Y

How to solve ALL duals optimally?

					opt
Day 1	$y_1 = 1$				1
Day 2	$y_1 = 1$	$y_2 = 1$			2
					⋮
Day $j\ (\le B)$	$y_1 = 1$	$y_2 = 1$	…	$\boxed{y_j = 1}$	j
Day $(B+1)$	$y_1 \ldots y_B = 1$		$y_{B+1} = 0$		B
Day $(B+2)$	$y_1 — y_B = 1$		$y_{B+1}, y_{B+2} = 0$		B

Primal Program

Optimize $\quad \min \left(\sum_{j=1}^{k} Z_j + x \cdot B \right)$

Subject to: $\quad Z_j \geqslant 0, \text{ for } j \in [1,k]$

$x \geqslant 0$

—

Dual Program

Optimize $\quad \max \left(\sum_{y \in Y} y \right)$

Subject to: $\quad y \in [0,1], \text{ for } y \in Y$

$\sum_{y \in Y} y \leqslant B$

Set $Y = \varnothing$

Day 1: Add new constraint: $Z_1 + x \geqslant 1$

Day 2: Add new constraint: $Z_2 + x \geqslant 1$

Day 1: Add new variable: y_1 to set Y $\quad y_1 = 1$

Day 2: Add new variable: y_2 to set Y $\quad y_2 = 1$

$\vdots$

$y_B = 1$

$y_{j(>B)} = 0$

(D) $D_{online} = D_{offline\text{-}opt}$

Solve all Duals optimally

Invariant (for each day): $\boxed{P \leqslant \alpha \cdot D}$ *

α- competitive

$P \leq \alpha \cdot (\text{opt-offline-dual-cost}) = \alpha \cdot (\text{opt-offline-primal-cost})$

$x \uparrow \quad Z_j \uparrow \quad \boxed{P = \alpha D}$

Primal Program

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

We can increase x if $P \not\leq \alpha D$

Subject to: $Z_j \geqslant 0$, for $j \in [1,k]$

$x \geqslant 0$

—

Dual Program

Optimize $\max\left(\sum_{y \in Y} y\right)$

Subject to: $y \in [0,1]$, for $y \in Y$

$\sum_{y \in Y} y \leqslant B$

Set $Y = \varnothing$

Day j ($\leqslant$ B): Add new constraint: $Z_j + x \geqslant 1$

$$x_{new} = x_{old} + \Delta x_j$$

$$Z_j = 1 - x_{new} = (1 - x_{old} - \Delta x_j)$$

Day j ($\leqslant$ B): Add new variable: y_j to set Y

Set $y_j = 1$

Dual cost $(j) = j$

Invariant (for each day): P $\leqslant$ $\alpha \cdot$ D (we will increase x_{new} till equality is obtained) $\boxed{P = \alpha D}$

$=$

$$\underbrace{\Delta P = \alpha \Delta D}_{} \Rightarrow \overbrace{(1 - x_{old} - \Delta x_j) + \Delta x_j B}^{\Delta P} = \alpha \cdot 1 \qquad \Delta x_j = \frac{\alpha - 1}{B - 1} + \frac{x_{old}}{B - 1}$$

$$\Delta P = (Z_j + \Delta x_j \cdot B)$$

$$\Delta x_j = \frac{\alpha - 1}{B - 1} + \frac{x_{old}}{B - 1}$$

$$x_{new} = \frac{\alpha - 1}{B - 1} + \frac{x_{old}}{B - 1} + x_{old}$$

$$= \frac{\alpha - 1}{B - 1} + x_{old}\left(\frac{B}{B - 1}\right)$$

$$x_{(0)} = 0$$

$$x_{(1)} = \frac{\alpha - 1}{B - 1}$$

$$x_{(2)} = \frac{\alpha - 1}{B - 1}\left(1 + \frac{B}{B - 1}\right)$$

$$x_{(j)} = \frac{\alpha - 1}{B - 1}\left(1 + \frac{B}{B - 1} + \left(\frac{B}{B - 1}\right)^2 + \cdots + \left(\frac{B}{B - 1}\right)^{j-1}\right) \qquad x_{(j)} = \alpha - 1\left(\left(\frac{B}{B - 1}\right)^j - 1\right)$$

Question: What is optimal α?

$\Delta D = 1$ for $j \leq B$

$\Delta D = 0$ for $j > B$

The relation $\Delta P = \alpha\Delta D = \alpha$ for $j \leqslant B$ days implies that

$$j \leq B \quad \left\{ \; x_{(j)} = \alpha - 1\left(\left(\frac{B}{B-1}\right)^{j} - 1\right) \right.$$

$P = \alpha D$

$\Delta P = 0$ after B^{th} day

After B^{th} day, $\Delta D = 0$ and thus ΔP must also be 0. Thus,

$(B+1)$

$(B+2)$

$$x_{(B)} = \alpha - 1\left(\left(\frac{B}{B-1}\right)^{B} - 1\right) \text{ must be } 1$$

$$\left(\sum_{j=1}^{k} z_j + x - B\right)$$

$$\alpha = 1 + \frac{1}{\left(\frac{B}{B-1}\right)^{B} - 1}$$

$z_{B+1} = z_{B+2} = 0$

$x_B = x_{B+1}$

$= x_{B+2}$

$\vdots$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = e \longrightarrow \quad \alpha = \frac{\left(\frac{B}{B-1}\right)^{B}}{\left(\frac{B}{B-1}\right)^{B} - 1} \approx \frac{e}{e-1}$$

$z_j + x \geq 1$

≈ 1.58

$\alpha(D) = 1$

$\alpha(P) = \frac{e}{e-1}$

The relation $\Delta P = \alpha \Delta D = \alpha$ for $j \leqslant B$ days implies that

$$x_{(j)} = \alpha - 1\left(\left(\frac{B}{B-1}\right)^j - 1\right)$$

Substituting value of α, we get

$$x_{(j)} = \frac{\left(\frac{B}{B-1}\right)^j - 1}{\left(\frac{B}{B-1}\right)^B - 1}$$

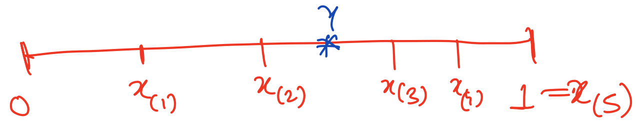
$D = D_{offline\text{-}opt} = P_{offline\text{-}opt}$

Theses values for $x(1), \cdots, x(B)$ ensures that on each day

$$\alpha = \frac{\left(\frac{B}{B-1}\right)^B - 1}{\left(\frac{B}{B-1}\right)^B} \approx \frac{e}{e-1}$$

$$P \leqslant \left(\frac{e}{e-1}\right)D = \left(\frac{e}{e-1}\right)P_{offline-opt}$$

Computing integral solution (from Primal LP)



B = 5

Online Primal (Det. sol^n)

$$x_{(1)} = 0.122 \qquad Z_1 = 0.878$$

$$x_{(2)} = 0.274 \qquad Z_2 = 0.726$$

$$x_{(3)} = 0.464 \qquad Z_3 = 0.536$$

$$x_{(4)} = 0.703 \qquad Z_4 = 0.297$$

$$x_{(5)} = 1 \qquad Z_5 = 0$$

$$\text{Prob}\left(\text{Buying upto day j}\right) = x_{(j)}$$

$$\text{Prob}\left(\text{Renting on day j}\right) = Z_j$$

H.W. 1

This will have higher expected cost

On each day toss a coin & decide

w.p. $x_{(j)} \longrightarrow$ Buy

$Z_{(j)} \longrightarrow$ Rent

Randomized algorithm

1. Choose a uniformly random γ in [0,1].
2. **If** (γ lies in $[x_{(i)}, x_{(i+1)}]$) **then** Buy on Day $(i+1)$
   Rent on Day $1, \ldots, i$

H. W. 2

expected cost of this algo $\leq \left(\frac{e}{e-1}\right) \cdot \min(k, B)$