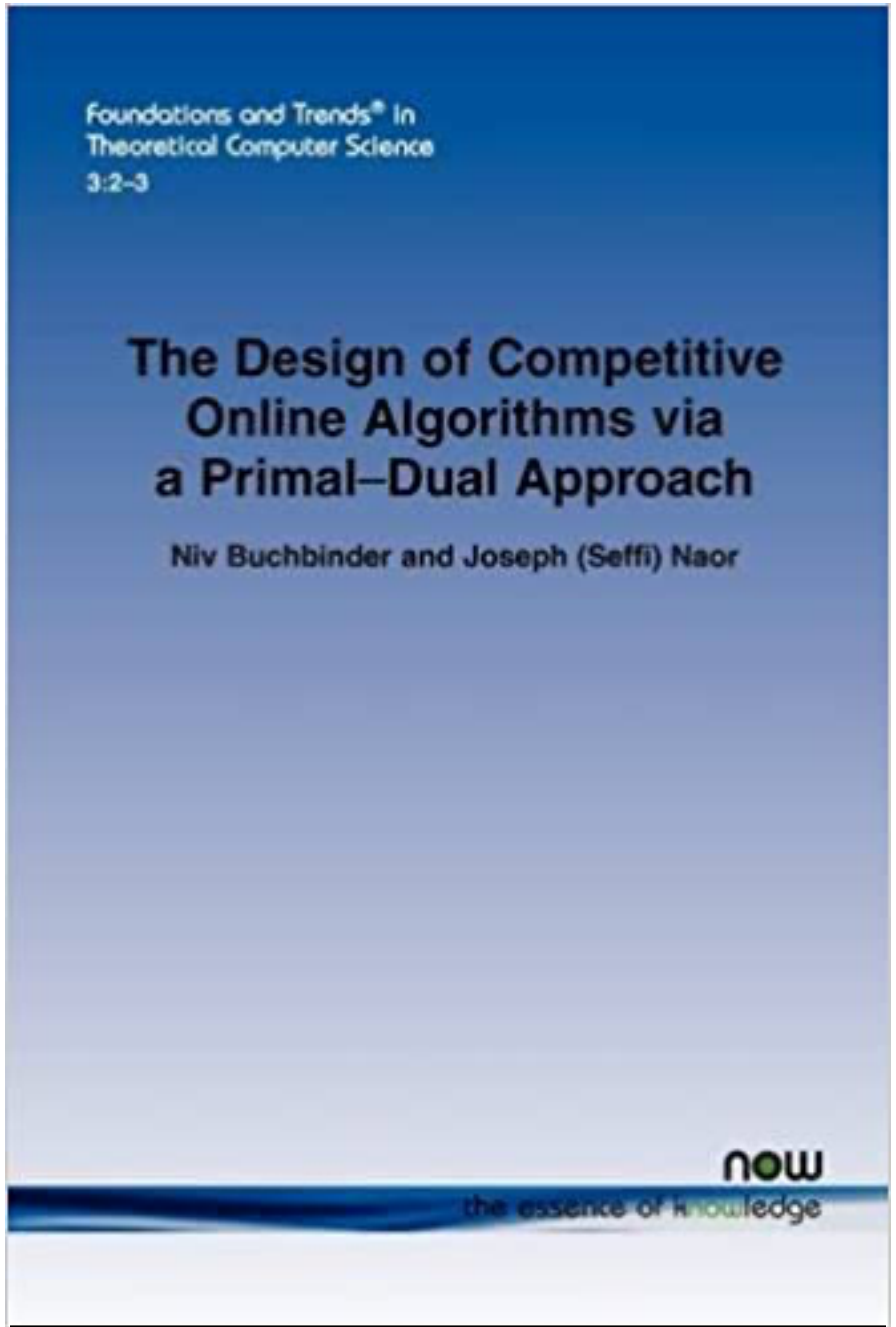
# COL 758: Advanced Algorithms

Lecture 32

# Reference Book




The Design of Competitive Online Algorithms via a Primal-Dual Approach:

Buchbinder and Naor

# “Online” Ski-Rental

**Ski Equipment:**

- Rent per day = 1
- Buy = B

$k$ = Number of days you will Ski (Unknown!)

Design an algorithm for which $\alpha = \max_{k \geq 1} \left( \frac{\text{cost paid by online algorithm}}{\text{opt cost if '}k\text{' was known}} \right)$ is minimised

## Primal Program

Optimize $\min\left(\sum_{j=1}^{k} Z_j + x \cdot B\right)$

Subject to: $Z_j \geqslant 0,\ \text{for } j \in [1,k]$

$x \geqslant 0$

—

## Dual Program

Optimize $\max\left(\sum_{y \in Y} y\right)$

Subject to: $y \in [0,1],\ \text{for } y \in Y$

$\sum_{y \in Y} y \leqslant B$

Set $Y = \varnothing$

**Day 1:** Add new constraint: $Z_1 + x \geqslant 1$

**Day 1:** Add new variable: $y_1$ to set $Y$

**Day 2:** Add new constraint: $Z_2 + x \geqslant 1$

**Day 2:** Add new variable: $y_2$ to set $Y$

Theorem: There is an online algorithm that solves all duals optimally, and ensures $\mathsf{P} \leqslant \alpha \cdot P_{offline-opt}$

$$\text{where, } \quad \alpha = \frac{\left(\frac{B}{B-1}\right)^B}{\left(\frac{B}{B-1}\right)^B - 1} \approx \left(\frac{e}{e-1}\right)$$

We have here for $j \in [0, B]$,
$$x_{(j)} = \frac{\left(\frac{B}{B-1}\right)^j - 1}{\left(\frac{B}{B-1}\right)^B - 1} \quad \text{and} \quad Z_j = 1 - x_{(j)}$$

# Computing integral solution (from Primal LP)

Randomized algorithm

1. Choose a uniformly random $\gamma$ in [0,1].
2. **If** ($\gamma$ lies in $[x_{(i-1)}, x_{(i)}]$) **then** Buy on day $i$, and Rent on days $1,\ldots,i-1$.

B = 5

| | |
|---|---|
| $x_{(1)} = 0.122$ | $Z_1 = 0.878$ |
| $x_{(2)} = 0.274$ | $Z_2 = 0.726$ |
| $x_{(3)} = 0.464$ | $Z_3 = 0.536$ |
| $x_{(4)} = 0.703$ | $Z_4 = 0.297$ |
| $x_{(5)} = 1$ | $Z_5 = 0$ |

**Claim 1(a):** $\text{Prob}\left[\text{Buying on day } i\right] = X_{(i)} - X_{(i-1)}$

**Claim 1(b):** $\text{Prob}\left[\text{Buying till day } i\right] = X_{(i)}$

**Claim 2:** $\text{Prob}\left[\text{Renting on day } i\right] = 1 - X_{(i)} = Z_i$

$$\textbf{Expected cost} = \sum_{i=1}^{k} \textbf{Prob}\left[\textbf{Renting on day } i\right] + \textbf{B} \cdot \textbf{Prob}\left[\textbf{Buying till day } i\right] = \sum_{i=1}^{k} Z_i + \textbf{B} \cdot X_{(k)} \leqslant \alpha \cdot P_{offline-opt}$$

# Ski-Rental Problem

**Ski Equipment:**

- Rent per day = 1
- Buy = B

$k$ = Number of days you will Ski (Unknown!)

Theorem: There is an online algorithm for Ski-Rental with expected competitive ratio

$$\alpha = \frac{\left(\frac{B}{B-1}\right)^B}{\left(\frac{B}{B-1}\right)^B - 1} \approx \left(\frac{e}{e-1}\right)$$

Bound is Tight !