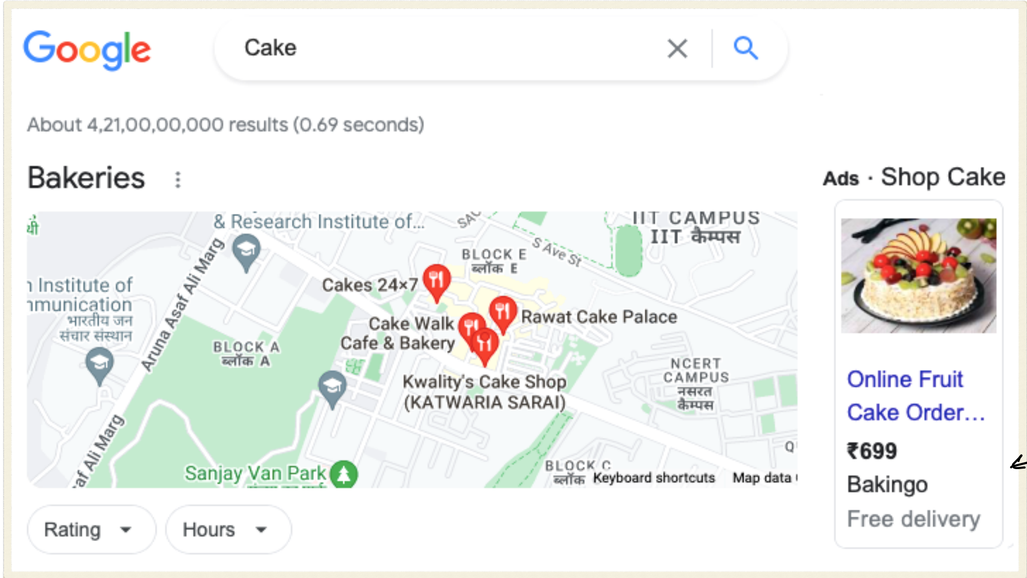
COL 758: Advanced Algorithms

Lecture 33

“Online” Ad-Allocation Problem

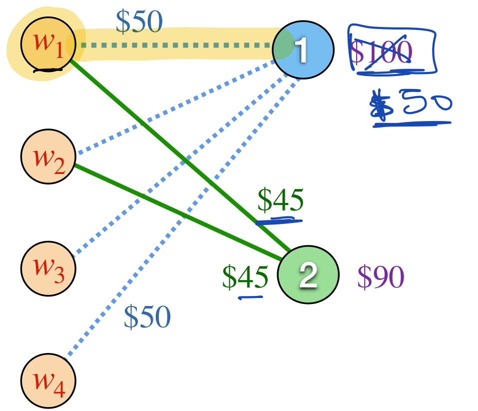


Ad slot

“Online” Ad-Allocation

Online: m search queries

n advertisers



Search Engine:

- Single Ad Slot
- Online sequence of search queries ($w_1, \ldots, w_m$)
- Advertisers $1, \ldots, n$ bid for the word w_j

  (different bids for different keyword search)
- One of the advertiser gets allocated for the ad slot for word w_j
- Advertisers has daily budget

Maximise = Total revenue generated from advertisers

Greedy: w_j – Allocate ad-slot the company which has largest bid

Revenue(Greedy) = 100 \$

Revenue(offline opt) = 190 \$

abhi easily close to $(200 - \varepsilon)$ \$

H.W.

$$\frac{1}{2} \leq \alpha(\text{greedy-algo}) \leq \frac{1}{2}$$

"Online" Ad-Allocation

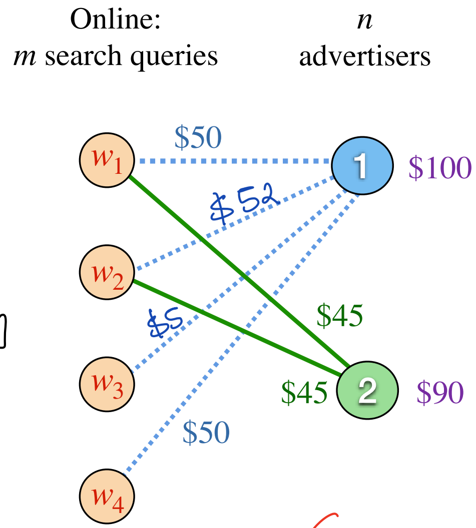
Search Engine:

- Single Ad Slot
- Online sequence of search queries $(w_1, \ldots, w_m)$
- Advertisers $1, \ldots, n$ bid for the word w_j

  (different bids for different keyword search)
- One of the advertiser gets allocated for the ad slot for word w_j $i \in [1, n]$
- Advertisers has daily budget

$1 - B_1$
$i - B_i$
$n - B_n$

Maximise = Total revenue generated from advertisers



$\left(1 - \frac{1}{e}\right)$

A deterministic algorithm for which $\alpha = \min_{\text{seq } \sigma} \left(\frac{\text{revenue generated in online algorithm}}{\text{opt revenue in offline setting}} \right)$ is maximized

Assumption = Bids are small compared to budget

Modeling Offline problem

"mn" variables

$w_j \leftarrow [1 \,...\, n]$

Integer Program

Optimize $\max\left(\sum_{i \in [1,n],\, j \in [1,m]} y_{ij}\, b_{ij}\right)$

Subject to:

$z_j \rightarrow \sum_{i=1}^{n} y_{ij} \leqslant 1$, for each word $j \in [1,m]$

$x_i \rightarrow \sum_{j=1}^{m} y_{ij} b_{ij} \leqslant B_i$, for each advertiser $i \in [1,n]$

$y_{ij} \geq 0 \leftarrow y_{ij} \in \{0,1\}$

b_{ij} = Bid for word j from bidder/advertiser i

$$y_{ij} = \begin{cases} 1 & \text{Bidder/Advertiser } i \text{ selected for word } j \\ 0 & \text{otherwise} \end{cases}$$

B_i = Budget of advertiser i

coeff in obj

primal variables

$$\frac{b_{ij}}{y_{ij}}$$

$$\begin{array}{lll} \text{word} & z_j & j \\ \text{adv.} & x_i & i \end{array} \left[\begin{array}{c} 1 \\ \hline b_{ij} \end{array}\right] \qquad \left[\begin{array}{c} 1 \\ \vdots \\ 1 \\ \hline B_1 \\ \vdots \\ B_n \end{array}\right]$$

$(m+n) \times (mn)$

Dual Obj $\min\left(\sum_{j=1}^{m} z_j + \sum_{i=1}^{n} x_i B_i\right)$

constraint (ij)

$\forall ij \qquad z_j + x_i b_{ij} \geq b_{ij}$

$\Updownarrow$

$\forall j \qquad z_j \geq \max_i \left(b_{ij}(1 - x_i)\right)$

Primal and Dual

$j \in \emptyset$
$j \in [1]$
$j \in [1,2]$
$j \in [1,2,3]$

m – is growing with time

1
2
3

Primal Program

Optimize $\max \left(\sum_{\substack{i \in [1,n], \\ j \in [1,m]}} y_{ij}\, b_{ij} \right)$

Subject to:

$$\sum_{i=1}^{n} y_{ij} \leqslant 1, \text{ for each word } j \in [1,m]$$

$$\sum_{j=1}^{m} y_{ij} b_{ij} \leqslant B_i, \text{ for each advertiser } i \in [1,n]$$

$$y_{ij} \geqslant 0$$

Dual Program

Optimize $\min \left(\sum_{i=1}^{n} B_i x_i + \sum_{j=1}^{m} z_j \right)$

Subject to:

$$z_j \geqslant \max_i \; b_{ij}\,(1 - x_i), \text{ for } j \in [1,m]$$

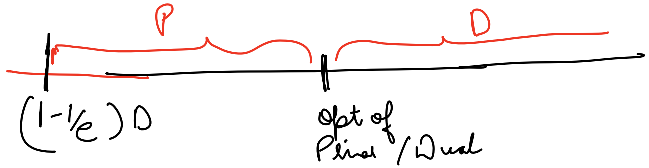
$$x_i \geqslant 0, \text{ for } i \in [1,n]$$

$$y_j \geqslant 0, \text{ for } j \in [1,m]$$

Question: What is relationship between a primal feasible P, and a dual feasible D?

$$\left(1 - \frac{1}{e}\right) D \leq P \leq D$$

$$(1 - 1/e)\ \text{opt-Primal} \leq P$$



Primal and Dual

Primal Program (I.P.)

Optimize $\max \left(\sum_{\substack{i \in [1,n],\\ j \in [1,m]}} y_{ij}\, b_{ij} \right)$

Subject to:

$\sum_{i=1}^{n} y_{ij} \leqslant 1$, for each word $j \in [1,m]$

$\sum_{j=1}^{m} y_{ij} b_{ij} \leqslant B_i$, for each advertiser $i \in [1,n]$

$y_{ij} \geqslant 0$ $\quad y_{ij} \in \{0,1\}$

Dual Program

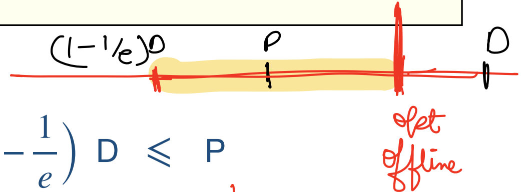
Optimize $\min \left(\sum_{i=1}^{n} B_i x_i + \sum_{j=1}^{m} z_j \right)$

Subject to:

$z_j \geqslant \max_i \; b_{ij}\,(1 - x_i)$, for $j \in [1,m]$

$x_i \geqslant 0$, for $i \in [1,n]$

$y_j \geqslant 0$, for $j \in [1,m]$



Invariant: A primal (*integer*) feasible P, and a dual feasible D, satisfying $\underbrace{\left(1 - \frac{1}{e}\right) \mathrm{D} \leqslant \mathrm{P}}$

To ensure this we will maintain $\left(1 - \frac{1}{e}\right) \Delta D \leqslant \Delta P$

P ≡ Primal Feasible

D ≡ Dual Feasible

Primal Program

Optimize $\max\left(\sum_{i\in[1,n],\ j\in[1,m]} y_{ij}\, b_{ij}\right)$

Subject to: $\sum_{i=1}^{n} y_{ij} \leqslant 1$, for each word $j \in [1,m]$

$\sum_{j=1}^{m} y_{ij}b_{ij} \leqslant B_i$, for each advertiser $i \in [1,n]$

$y_{ij} \geqslant 0$

fractional

Dual Program

Optimize $\min\left(\sum_{i=1}^{n} B_i x_i + \sum_{j=1}^{m} z_j\right)$

Subject to: $z_j \geqslant \max_i\, b_{ij}\,\underline{(1-x_i)}$, for $j \in [1,m]$

$x_i \geqslant 0$, for $i \in [1,n]$

$y_j \geqslant 0$, for $j \in [1,m]$

$1 - x_i = 0$ iff bidder i has exhausted all budget

Beginning: Initialise all primal/dual variables to 0

$\beta \cdot D \leq P$

$\boxed{\beta \cdot \Delta D \leq \Delta P}$

Word j arrives:

Let i denote the index maximising $\left(b_{ij}\,(1 - x_i)\right)$

Assign word j to bidder 'i', i.e., set $y_{ij} = 1$

$z_j = b_{ij}(1 - \underset{\text{old } x_i}{x_i})$, and $\underset{\text{new } x_i}{x_i} = \underset{\text{old } x_i}{x_i} + \Delta x_i$

x_i : intuitively indicates the fraction of budget bidder i has spent

$1 - x_i$: intuitively indicates the fraction of budget bidder i is left with

Bidding decision function of:
- Bids, and
- $1 - x_i$

What happens if we don't take in account $(1 - x_i)$?

Primal Program

Optimize $\max \left(\sum_{\substack{i \in [1,n], \\ j \in [1,m]}} y_{ij}\, b_{ij} \right)$

Subject to: $\sum_{i=1}^{n} y_{ij} \leqslant 1$, for each word $j \in [1,m]$

$\sum_{j=1}^{m} y_{ij} b_{ij} \leqslant B_i$, for each advertiser $i \in [1,n]$

$y_{ij} \geqslant 0$

Dual Program

Optimize $\min \left(\sum_{i=1}^{n} B_i x_i + \sum_{j=1}^{m} z_j \right)$

Subject to:

$z_j \geqslant \max_i \; b_{ij} \,(1 - x_i)$, for $j \in [1,m]$

$x_i \geqslant 0$, for $i \in [1,n]$

$y_j \geqslant 0$, for $j \in [1,m]$

Beginning: Initialise all primal/dual variables to 0

Word j arrives:

Let i denote the index maximising $\left(b_{ij}\,(1 - x_i)\right)$

Assign word j to bidder 'i', i.e., set $y_{ij} = 1$

$z_j = b_{ij}(1 - x_i)$, and $x_i = x_i + \Delta x_i$

$\Delta P = b_{ij}$

$\Delta D = b_{ij}(1 - x_i) + B_i \Delta x_i$

$b_{ij} = \beta \left(b_{ij} (1 - x_i) + B_i \Delta x_i \right)$

We require $\Delta P = \beta \cdot \Delta D$, where β is fractional.

Word j arrives:

We require $\Delta P = \beta \cdot \Delta D$, that is $b_{ij} = \beta\left(b_{ij}(1 - x_i) + B_i \Delta x_i\right)$.

Solving this gives,

$$\Delta x_i = \frac{b_{ij}}{B_i} x_i + \frac{b_{ij}}{B_i}\left(\frac{1-\beta}{\beta}\right)$$

$$x_i = \left(1 + \frac{b_{ij}}{B_i}\right) x_i + \frac{b_{ij}}{B_i}\left(\frac{1-\beta}{\beta}\right)$$ $\longleftarrow$ increasing exponentially bcoz we want $\Delta P = \beta \cdot \Delta D$

Suppose bidder i gets ad slot for words $j_1 \cdots j_q$, & $b_{ij_1} + b_{ij_2} + \cdots + b_{ij_q} = B_i$

$$x_{i(0)} = 0$$

$$x_{i(j_1)} = \left(\frac{1-\beta}{\beta}\right)\left(\frac{b_{ij_1}}{B_i}\right)$$

$$\beta = \left(1 - \frac{1}{e}\right)$$

$$x_{i(j_1 \cdots j_q)} = \left(\frac{1-\beta}{\beta}\right)\left[\left(1+\frac{b_{ij_1}}{B_i}\right)\left(1+\frac{b_{ij_2}}{B_i}\right)\cdots\left(1+\frac{b_{ij_q}}{B_i}\right) - 1\right] \approx \underbrace{\left(\frac{1-\beta}{\beta}\right)(e-1)}_{=1}$$

$$\boxed{\text{CLAIM: If } \gamma_1 + \gamma_2 + \cdots + \gamma_Q = Y,\ \gamma_i \ll Y \text{ then } \prod_{i=1}^{Q}\left(1+\frac{\gamma_i}{Y}\right) \approx e \quad \left(\text{All } \gamma_i = 1 \quad \left(1+\frac{1}{n}\right)^n \approx e\right)}$$

$$\overset{\beta}{\overset{\|}{\left(1-\frac{1}{e}\right)}} \Delta D \leq \Delta P_{int}$$

$$\left(1-\frac{1}{e}\right) D \leq P_{int}$$

$$\left(1-\frac{1}{e}\right) P_{ideal\text{-}opt} \leq P_{int}$$