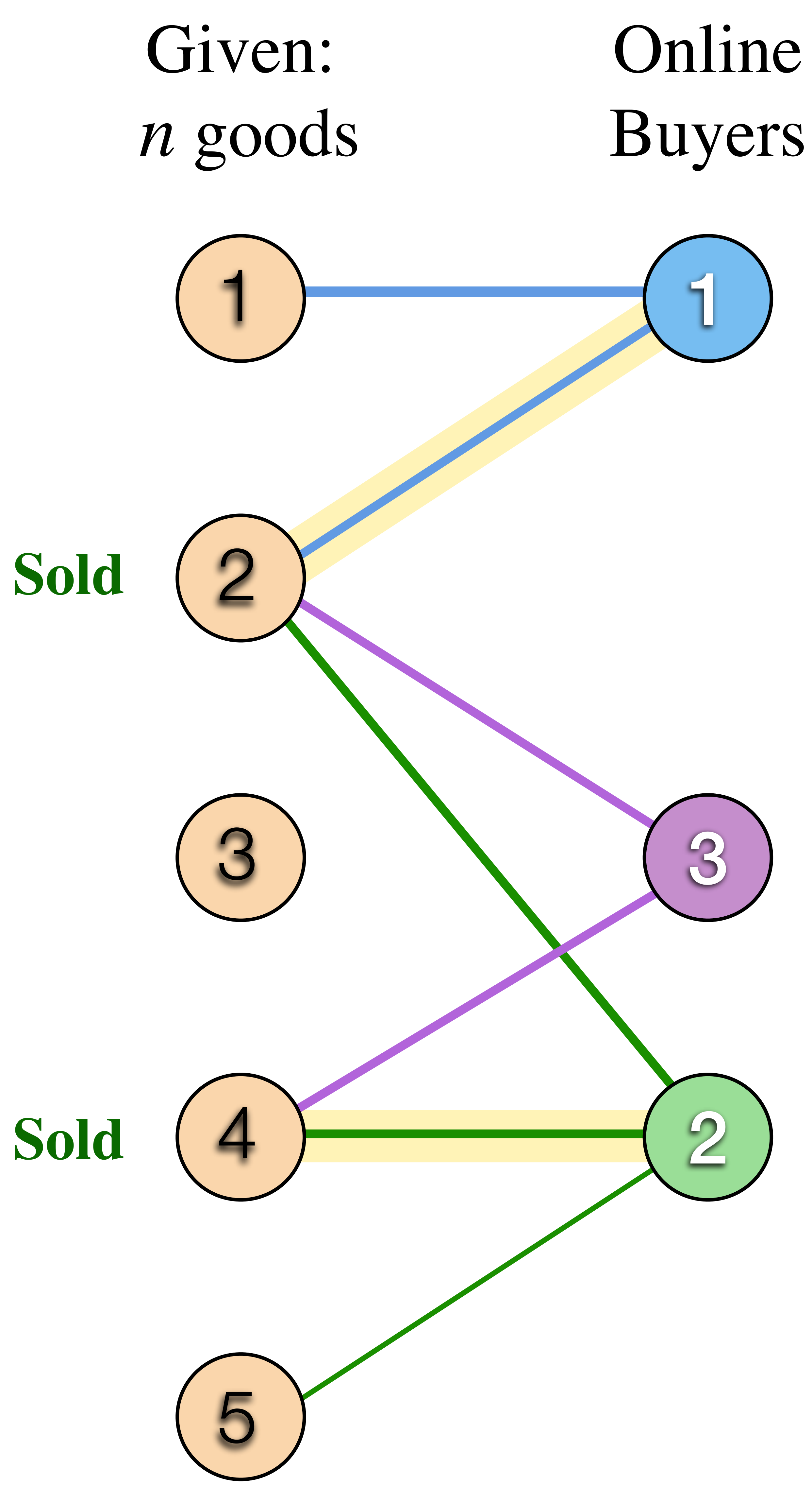
COL 758: Advanced Algorithms

Lecture 36

Online Bi-partite Matching



Preferences of (online) buyers $1, \ldots, m$ not known apriori.

$$\alpha = \frac{\text{size of online matching}}{\text{size of optimal offline matching}}$$

Lecture 28: A deterministic algorithm with competitive ratio $(1/2)$.

Todays' Lecture: A strategy to maintain a matching with competitive ratio $(1 - 1/e)$.

Randomized Algorithm (Lec 29)

Initialization: $I_{left} \leftarrow [1,n]$

When Buyer j arrives with choices C_j :

If $(I_{left} \cap C_j)$ is non empty, then

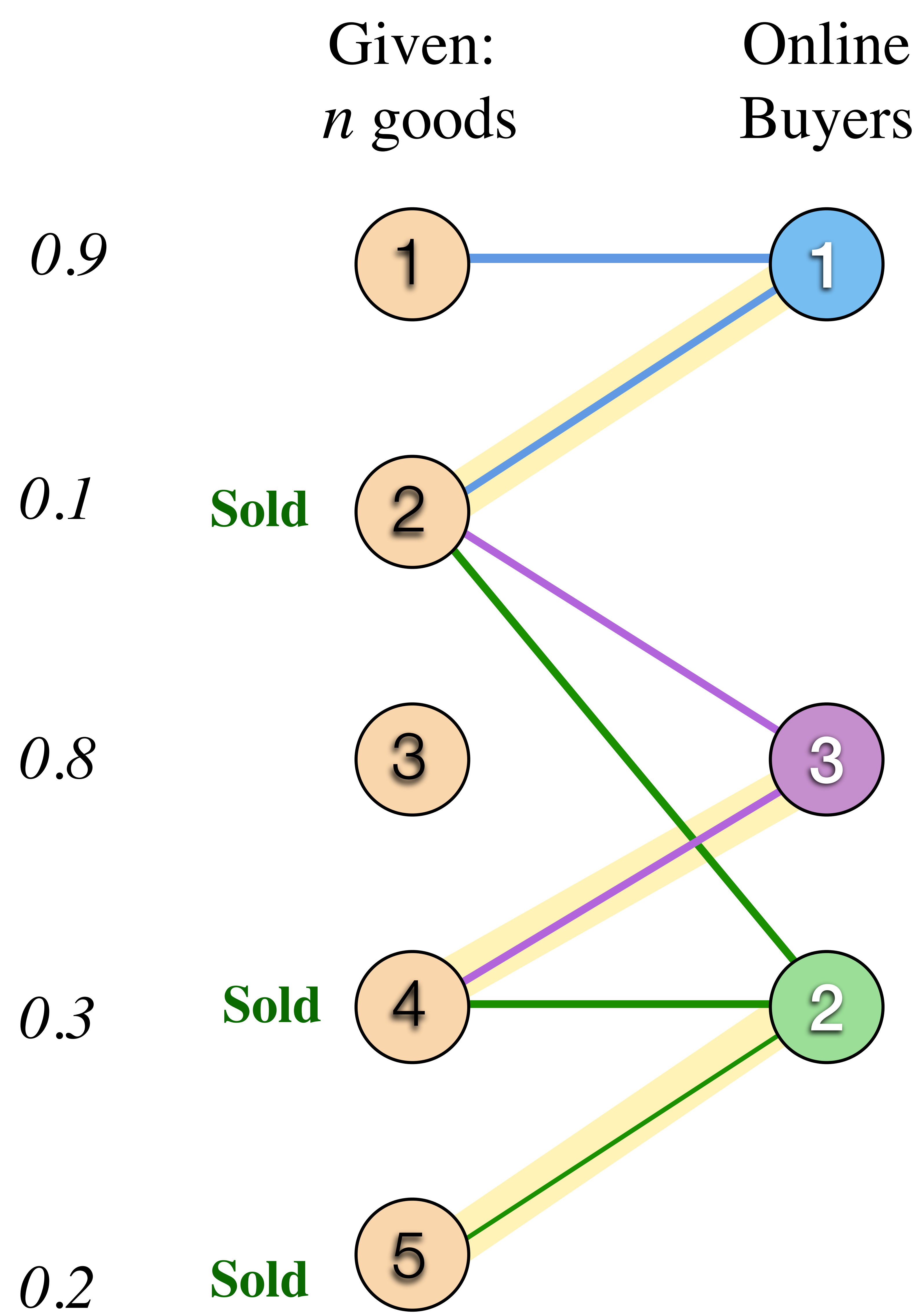
- Let i_0 be a randomly chosen item in set $(I_{left} \cap C_j)$.
- Match Buyer j to good i_0
- Remove i_0 from I_{left}.

Question: What is competitive ratio of this algorithm?

Ans: The competitive ratio α is half as we can show:

- $\alpha \geqslant 1/2$ (size of maximal matching is at least half the size of maximum matching)
- $\alpha \leqslant 1/2$ (counter-example?)

Randomized Algorithm: Main Idea



Associate random prices with all items(goods)

Match buyer *j* to good with least price

New Randomized Algorithm

Initialization:

$I_{left} \leftarrow [1,n]$

For $(1 \leqslant i \leqslant n)$: $X_i = U[0,1]$ be random variable. Set price of item i as $p_i = e^{X_i - 1}$ $(p_i \in [e^{-1},1])$

When Buyer j arrives with choices C_j :

If $(I_{left} \cap C_j)$ is non empty, then

- Let i_0 be item in set $(I_{left} \cap C_j)$ with "*least price*".
- Match buyer j to good i_0 .
- Remove i_0 from I_{left} .

Random Variables: Revenue and Utility

Revenue from item i:

$$R_i = \begin{cases} p_i & \text{if item } i \text{ is sold,} \\ 0 & \text{otherwise.} \end{cases}$$

Utility of Buyer j:

$$U_j = \begin{cases} 1 - p_i & \text{if } j \text{ buys item } i, \\ 0 & \text{otherwise.} \end{cases}$$

Claim 1: The sum $\left(\sum_{i=1}^{n} R_i + \sum_{j=1}^{m} U_j \right)$ is the size of matching obtained by our algorithm.

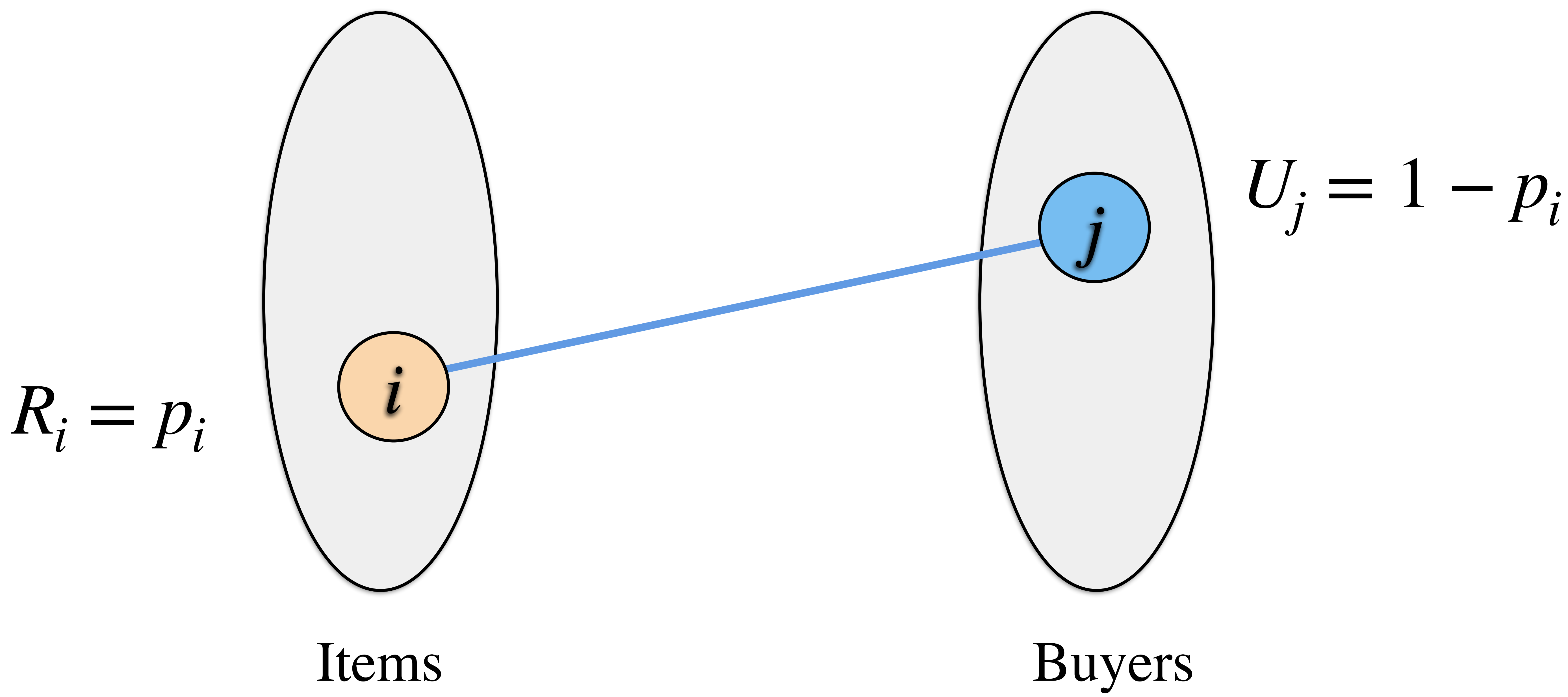
Proof:

For each edge (i, j), we have

$R_i + U_j = 1$

if and only if

(i, j) lies in matching



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Claim 1: The sum $\left(\sum_{i=1}^{n} R_i + \sum_{j=1}^{m} U_j \right)$ is the size of matching obtained by our algorithm.

Implication:

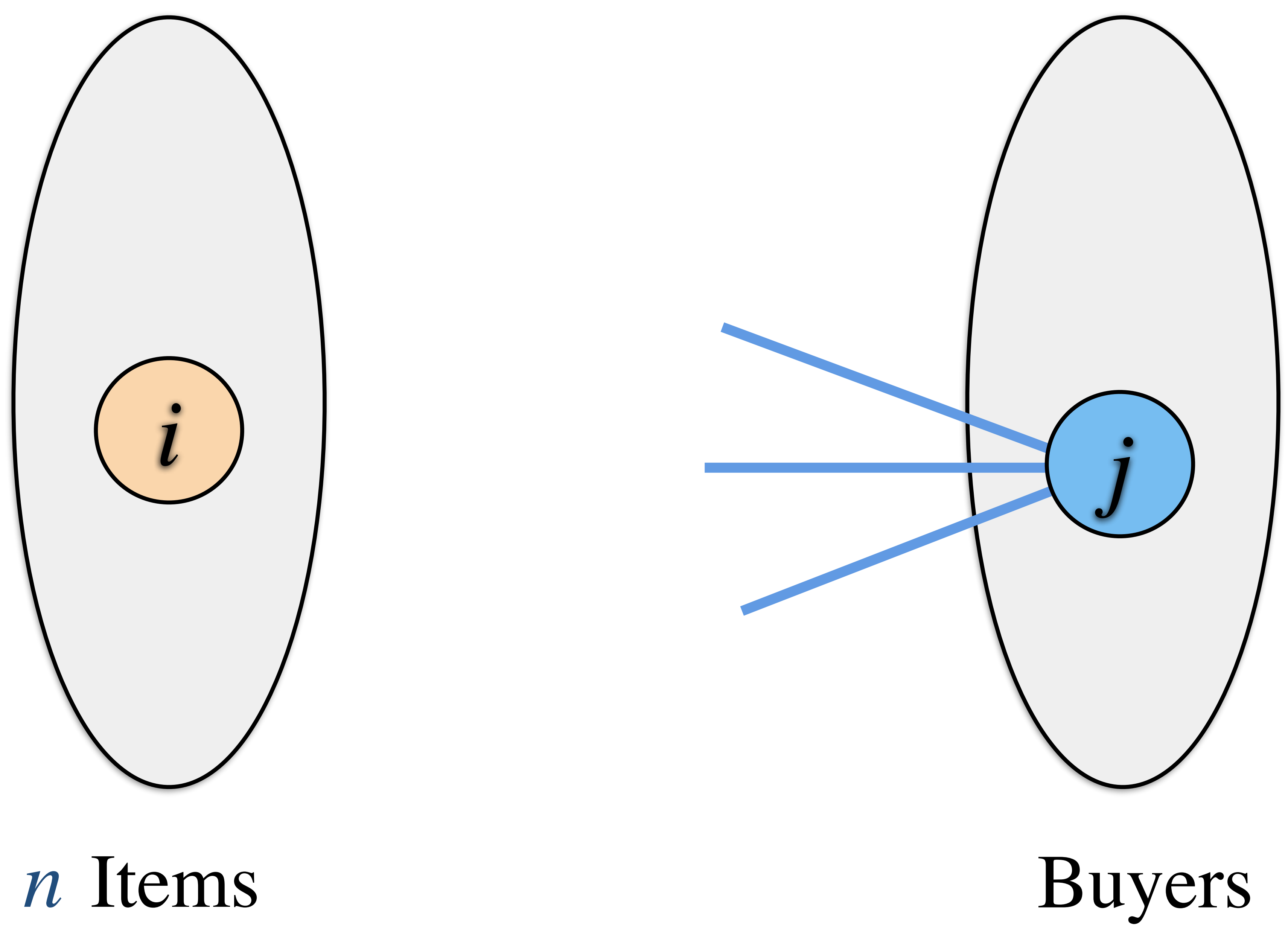
If M^* = optimal offline maximum matching, and M = matching obtained by our algorithm, then

$$E[\,|M|\,] = \left(\sum_{i=1}^{n} E[R_i] + \sum_{j=1}^{m} E[U_j] \right) \geqslant \sum_{(i,j)\in M^*} \underbrace{E[R_i + U_j] \geqslant \left(1 - \frac{1}{e}\right) |M^*|}$$

We will show that for any edge (i, j),
$E[R_i + U_j] \geqslant (1 - 1/e)$

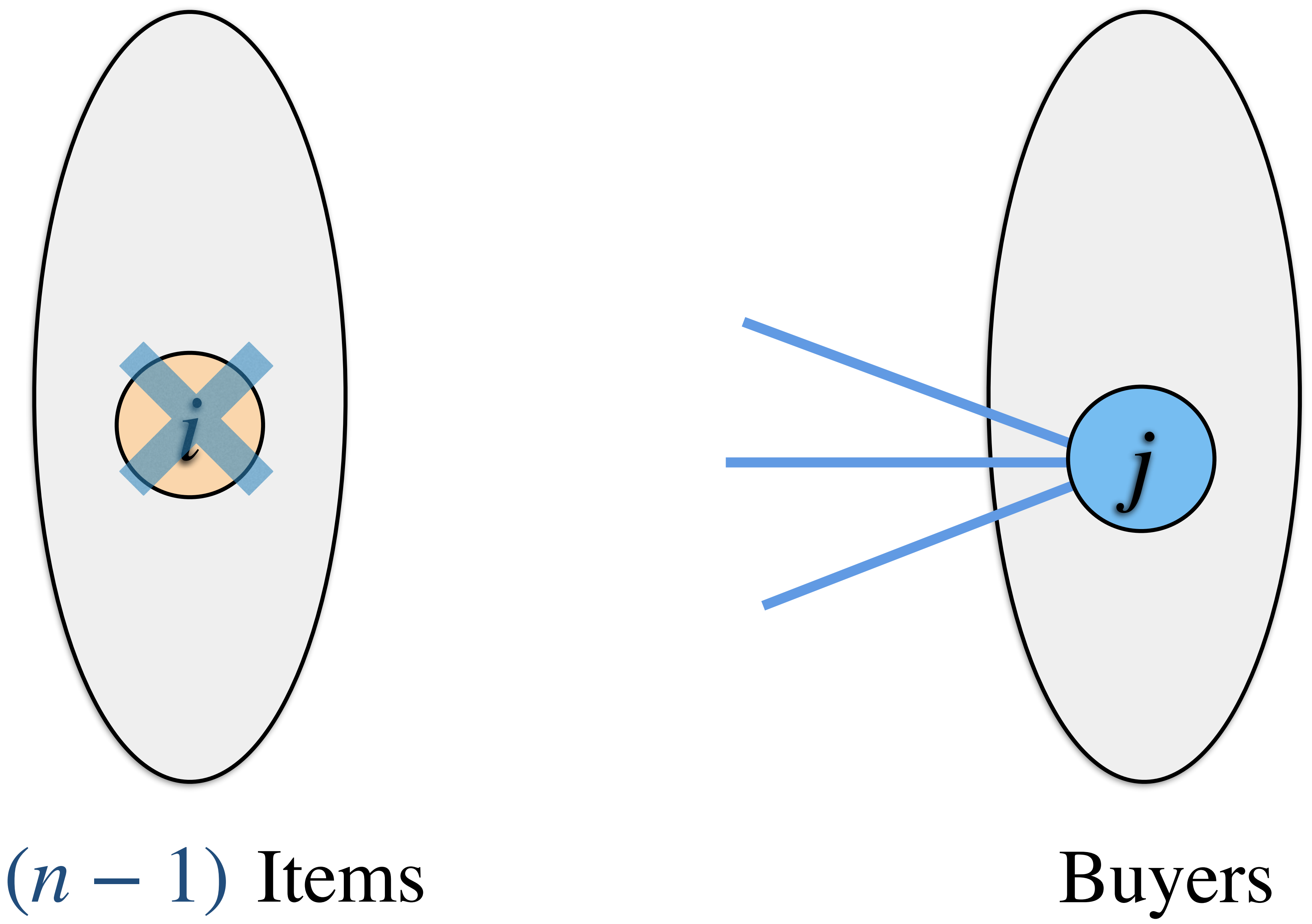
Consider a choice P_{-i} of prices for all items other than i. We will show $E\left[(R_i + U_j)\,\middle|\,P_{-i}\right] \;\geq\; \left(1 - \frac{1}{e}\right)$

Graph G:
Define G to be the graph with
— all n items, and
—all m buyers
Price evaluations of item other than i^{th} item are as vector P_{-i}



$E\left[U_j\,\middle|\,P_{-i}\right]$

Graph G':
Define G' to be the graph with
— all items other than i, and
— all m buyers
Price evaluations of all items are as per vector P_{-i}



U_j' = utility of buyer j in graph G'.

Consider a choice P_{-i} of prices for all items other than i. We will show $E\left[(R_i + U_j) \middle| P_{-i}\right] \geq \left(1 - \frac{1}{e}\right)$

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Graph G':
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Claim 2: Let U'_j be utility of buyer j in graph G'. Then $E\left[U_j \middle| P_{-i}\right] \geqslant U'_j$.

Proof:

Price of item sold to buyer j in G $\leqslant$ Price of item sold to buyer j in G'

Consider a choice P_{-i} of prices for all items other than i. We will show $E\left[(R_i + U_j)\middle|P_{-i}\right] \geq \left(1 - \frac{1}{e}\right)$

Graph G:
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Graph G':
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Claim 3: Let q be such that $U_j' = 1 - q$. Then, $E\left[R_i\middle|P_{-i}\right] \geqslant q - 1/e$.

Proof: Here q = amount buyer j is paying in G'. So, if $p_i < q$ then item 'i' must be sold in graph G!!

(Recall: $p_i = e^{X_i - 1} < q$ if and only if $X_i < 1 + \ln q$).

$$E\left[R_i\middle|P_{-i}\right] \geqslant \int_0^{1+\ln q} e^{X_i-1}\, d(X_i) = \left[e^{X_i-1}\right]_0^{1+\ln q} = q - 1/e$$

Random Variables: Revenue and Utility

Claim 2: Let U'_j be utility of buyer j in graph G'. Then $E\left[U_j \middle| P_{-i}\right] \geqslant U'_j$.

Claim 3: Let q be such that $U'_j = 1 - q$. Then, $E\left[R_i \middle| P_{-i}\right] \geqslant q - 1/e$.

These two together implies

Claim 4: For any edge (i, j), $E[R_i + U_j]$ is at least $(1 - 1/e)$.