$n = |V|$

$m = |E|$

Graphs

- Static
- Changing with time
  - Dynamic (constantly evolving) (Facebook network)
  - Fault-Tolerant (small changes) (Rail network over small period)

Data Structure for graphs: (e.g.)

① Sparsifying a graph to preserve some properties (connectivity → spanning forest)

② Graph Algorithms (eg. BFS/DFS) for networks changing with time.

③ Oracles: Distance Oracle

Size $= O(n^2)$

Time $= O(1)$

$x, y \rightarrow \boxed{\text{Distance Matrix}} \rightarrow dist_G(x,y)$

Def^n Spanners:

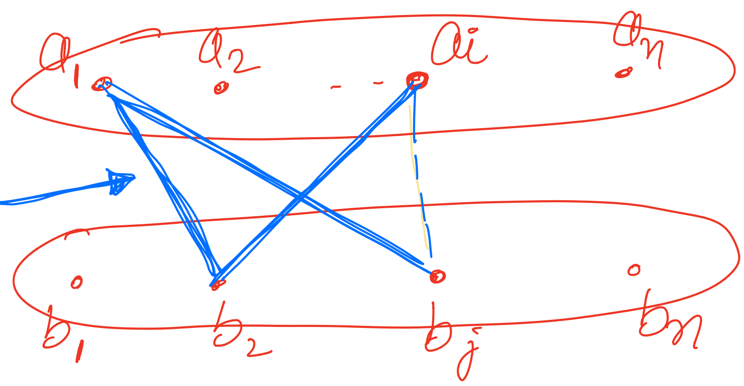
Given a graph $G=(V,E)$, a subgraph $H=(V, E_H \subseteq E)$ is said to be an α- multiplicative spanner of G if $\forall\, x,y \in V(G)$

$$dist_H(x,y) \leq \alpha \cdot dist_G(x,y)$$

Ques: Can you have $|E_H| = o(n^2)$ & $\alpha \lesssim 3$?

Ans: No, a simple counter-example is $K_{\frac{n}{2},\frac{n}{2}}$ on n vertices and $\frac{n^2}{4}$ edges

New path of length 3 from a_i to b_j in $G \setminus (a_i, b_j)$



Orig dist a_i to b_j $= 1$

So, stretch is "3" even when H has $\left(\frac{n^2}{4} - 1\right)$ edges.

⊛ We will give construction of $(2k-1)$ multiplicative spanner of stretch $= 2k-1$ and size $= O(n^{1+1/k})$.

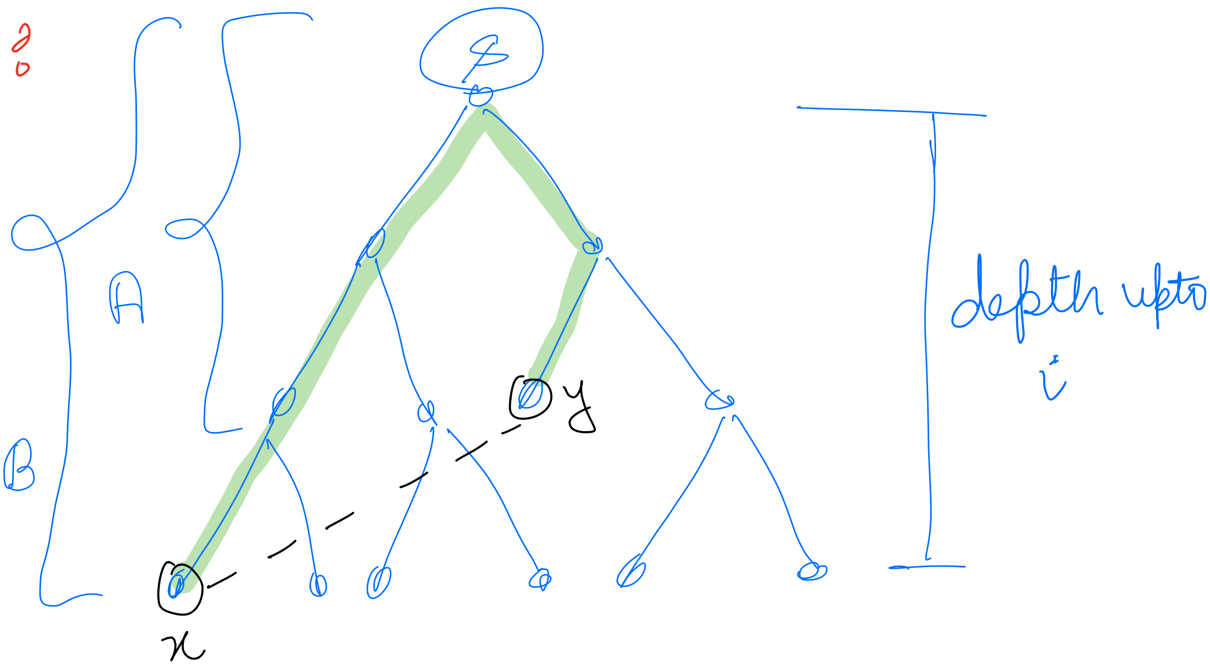
(α) stretch $= 2k-1$		size $= O(n^{1+1/k})$
$k=2$	3	$O(n^{3/2})$
$k=3$	5	$O(n^{4/3})$
⋮	⋮	⋮

For $k = \log_2 n$ $\quad n^{1+\frac{1}{k}} = (2^k)^{1+\frac{1}{k}} = 2^k \cdot 2 = 2n$

Defⁿ: An edge (x,y) will said to be "Handled" w.r.t. H if $d_H(x,y) \le 2k-1$.

Main Idea:

Partial BFS tree upto depth i:



If we add edges of this tree to our spanner "H" then for any two vertices $x, y \in B$ such that either x or y lie in A we have,

$$\text{dist}_H(x,y) = 2i-1.$$

Remark: $B = A$ if no vertices at depth $= i$

Ques: Can we say all edges (x,y) incident to at atleast one endpoint of A are handled?

Ans: Yes, if $i \leq k$, because neighbors of A must lie in B

To ensure average degree $\leq n^{1/k}$ we grow tree only upto that index "i" for which

$$\frac{|B|}{|A|} \leq n^{1/k} \text{ because:}$$

no. of edges added is "$|B|-1$"

and, vertices whose all incident edges are handled is "$|A|$"

Avg no of edges added per vertex in A $\dfrac{|B|-1}{|A|} \leq n^{1/k}$.

Notation:

$G[U] \equiv$ For a set U of vertices, $G[U]$ is subgraph of G with vertex-set U, and edge-set $\stackrel{\triangle}{=} E(G) \cap (U \times U)$

Algorithm:

(denotes set of unmarked vertices)

① Let $U = \{v_1, v_2, \ldots, v_n\}$ be vertices of G initially all set as unmarked.

② Initialize $H = (V, E_H = \emptyset)$

③ For $j = 1$ to n: { If v_j is unmarked: Generate Tree(v_j)}

Generate Tree(v):

① Compute smallest integer $i \geq 1$ s.t.

$$\left|\underbrace{BFS(v, depth \leq i, G[U])}_{}\right| \leq n^{1/k}. \qquad \text{—(Eq 1)}$$

BFS tree of V in $G[U]$ upto depth i

② Add edges$\left(\overbrace{BFS(v, depth \leq i, G[U]}^{}\right)$ to E_H.

③ Set vertices$\left(BFS(v, depth \leq i-1, G[U]\right)$ as marked, & remove them from U.

Lemma 1: In process Generate tree (v) if we remove α vertices from U, then at most $(n^{1/k})(\alpha)$ edges are added to E_H.

Proof: No of vertices removed from U is $\alpha = |BFS(v, depth \leq i-1, G[U])|$

No of edges added to E_H is $|BFS(v, depth \leq i, G[U])| - 1 \leq n^{i/k} - 1$

By def^n of i

CASE 1: $i = 1$	CASE 2: $i > 1$
If $i=1$, then $\alpha=1$ as only v_i is removed from U. No of edges added $\leq n^{i/k} - 1$ i.e. $\leq n^{1/k} - 1$ (as $i=1$) $\therefore$ no. of edges added to E_H is at most $\alpha \cdot n^{1/k}$	If $i>1$, then $\alpha \geq n^{\frac{i-1}{k}}$ (by def^n of i). No of edges added to E_H is $n^{i/k} - 1 \leq n^{1/k} \cdot n^{\frac{i-1}{k}} \leq \alpha \cdot n^{1/k}$ $\therefore$ no of edges added to E_H is at most $\alpha n^{1/k}$

$\square$

Homework: Prove that above lemma shows H has $O(n^{1+1/k})$ edges.

Lemma 2: In procedure GenerateTree(v), index $i \leq k$.

Proof:
Note $|BFS(v, \text{depth} \leq k, G[U])| \leq |V(G)| = n = n^{k/k}$
As Eq. 1 satisfied for index k, i is $\leq k$.

$\square$

Lemma 3: After execution of Algorithm all the edges of graph G are handled.

Proof: Consider edge (x,y) in G.

Assumption: Without loss of generality assume x is removed at some time or before y from set U.

v ← Let v be s.t. x is removed from U during execution of procedure GenerateTree(v)

$\Rightarrow x, y$ both belonged to $G[U]$ before execution of procedure GenerateTree(v)

As $x \in BFS(v, \text{depth} \leq i-1, G[U])$ and $y \in N(x)$,

we have, $y \in BFS(v, \text{depth} \leq i, G[U])$

So, $d_{G[U]}(x, v) \leq i-1$, and $d_{G[U]}(y, v) \leq i$

After adding to E_H edges of $BFS(v, \text{depth} \leq i, G[U])$, we get

$$
\begin{aligned}
d_H(x,y) &\leq d_H(x, v) + d_H(v, y) \\
&= d_G(x, v) + d_G(v, y) \\
&\leq 2i - 1 \\
&\leq 2k - 1
\end{aligned}
$$

H.W. Time Complexity $= O(m+n)$

Hint: Show that for a vertex x we spent only $O(\deg(x))$ time

Ques. What is one major drawback of this algorithm?

Works for undirected unweigthed graphs only.

Ques. Can such a result exists for digraphs?

No, example take "directed bipartite graph."

Scenario of weighted undirected graphs

Algorithm

① Initialize $H = (V, E_H = \emptyset)$

② Let $(e_1 \dots e_m)$ be edges in G s.t.

$wt(e_1) \le wt(e_2) \le \dots \le wt(e_m)$

③ For $i = 1$ to m:

let $x, y \leftarrow$ endpoints of e

If $\left(\frac{dist_H(x,y)}{wt(e)} > 2k-1 \right)$: Add e to E_H

(Correctness)

Claim 1: $\forall\ x, y \in V$, $dist_H(x,y) \le (2k-1)\, dist_G(x,y)$

Proof Hint: For each edge $e = (x,y)$ in G,

$dist_H(x,y) \le (2k-1) \cdot wt(e) \le (2k-1) \cdot dist_G(x,y)$

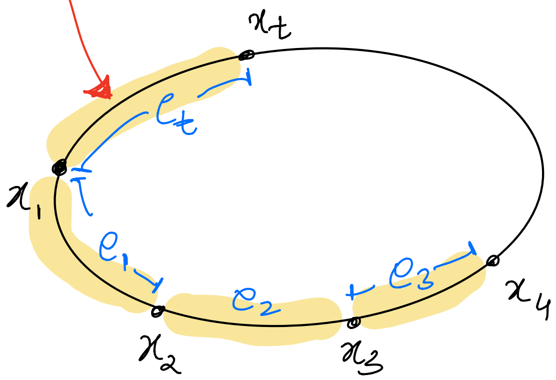
So all edges in any shortest path stretched by 2k-1 factor!

Claim 2: Each cycle in H has length $\ngeq 2k$.

suppose this is last edge added to H

Proof:

Assume there



cycle of size $\leq 2k$

$H_0 \leftarrow$ subgraph of H just before adding e_t.

$\Rightarrow e_1 \; e_2 \; \ldots \; e_{t-1} \in H_0$

why?

As $t \leq 2k \Rightarrow wt_H(x_1 \to x_2 \to \ldots \to x_t) \leq (2k-1)\, wt(e_t)$

As $e_t \in H$, we have $\dfrac{dist_H(x_t, x_1)}{wt(e_t)} \ngeq 2k-1$

which is a contradiction.

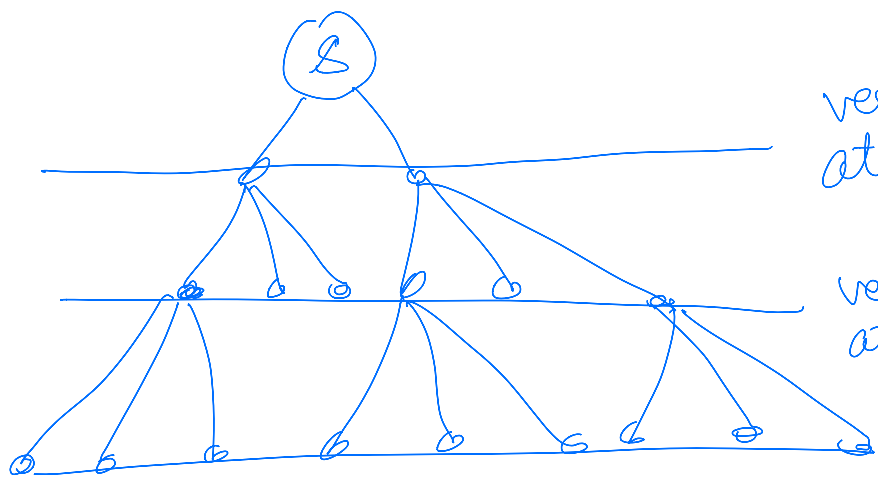
Girth (G) = length of smallest cycle

Claim 3 : For any n vertex graph "G" with girth $\geq 2k+1$, $E(G) = O(n^{1+1/k})$

Proof:

(*) First repeatedly remove vertices (with its incident edges) of deg $\leq n^{1/k}$.

(*) Let $\tilde{G}$ be left over graph. If $\tilde{G} \neq$ NULL, take an s and compute BFS(s, level $\leq k$)



vertices at lv=1 $\geq n^{1/k}$

vertices at lv=2 $\geq n^{2/k}$

vertices at lv=3 $\geq n^{3/k}$

Sum total vertices $\geq 1 + n^{1/k} + n^{2/k} + \dots + n^{k/k}$
$\geq n$.

Hence, we get a contradiction and thus $\tilde{G}$ must be empty.