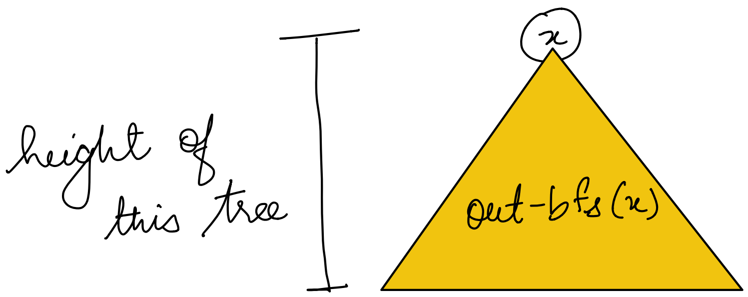
Looking at 2 problems in directed graphs:

(i) 2-approximation of eccentricities

$$ecc_G(x) = \max_{y \in V(G)} d_G(x,y)$$

For undirected graphs it is:



(ii) Finding eccentricity spanner of stretch = 2

A sparse subgraph $H = (V, E_H)$ satisfying

$$ecc_H(x) \leq 2 \cdot ecc_G(x)$$

$\forall x \in V(G)$.

$$diam(G) = \max_{x,y \in V(G)} d_G(x,y) = \max_{x \in V(G)} ecc_G(x)$$

$$centre(G) = \arg\min_{x} ecc_G(x)$$

$$radius(G) = \min_{x} ecc_G(x) = ecc(\text{centre of } G)$$

Recall def^n of spanner of G:

A subgraph $H = (V, E_H)$ of G s.t.

$\forall x, y \in V(G) \quad d_H(x,y) \le k \cdot d_G(x,y)$

stretch

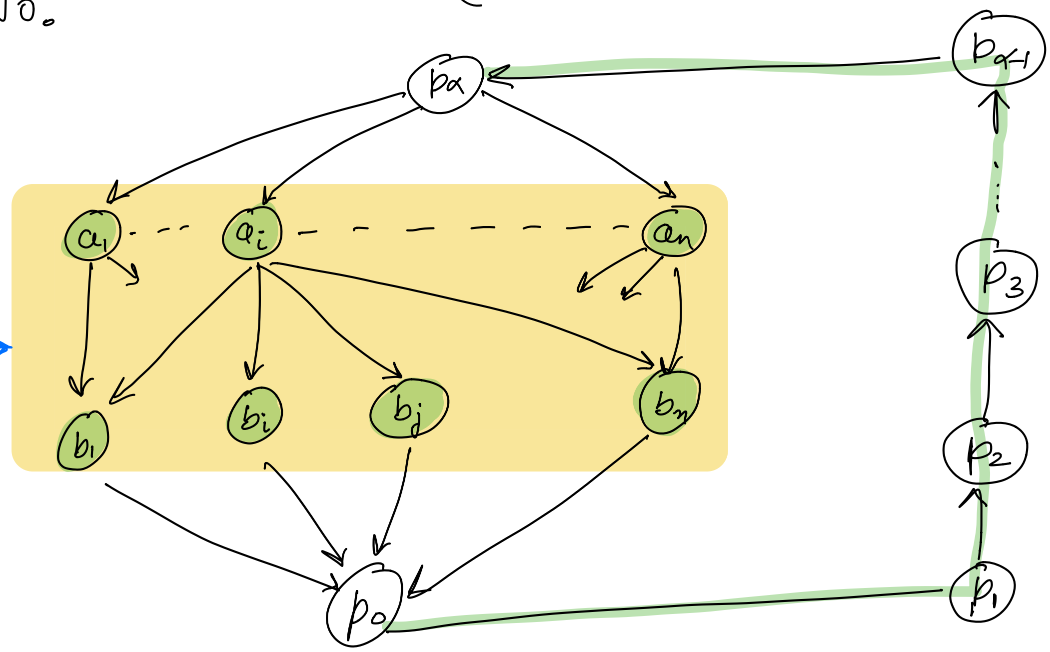
$\Rightarrow$ A k-distance spanner is also k-eccentricity spanner.

Ques Can we have sparse distance-spanner for strongly connected graphs?

$(\alpha \le n$, so $|V(G)| = \Theta(n))$

Sol^n: No.

complete directed bipartite with n^2 edges



$diam(G) = \Theta(\alpha)$

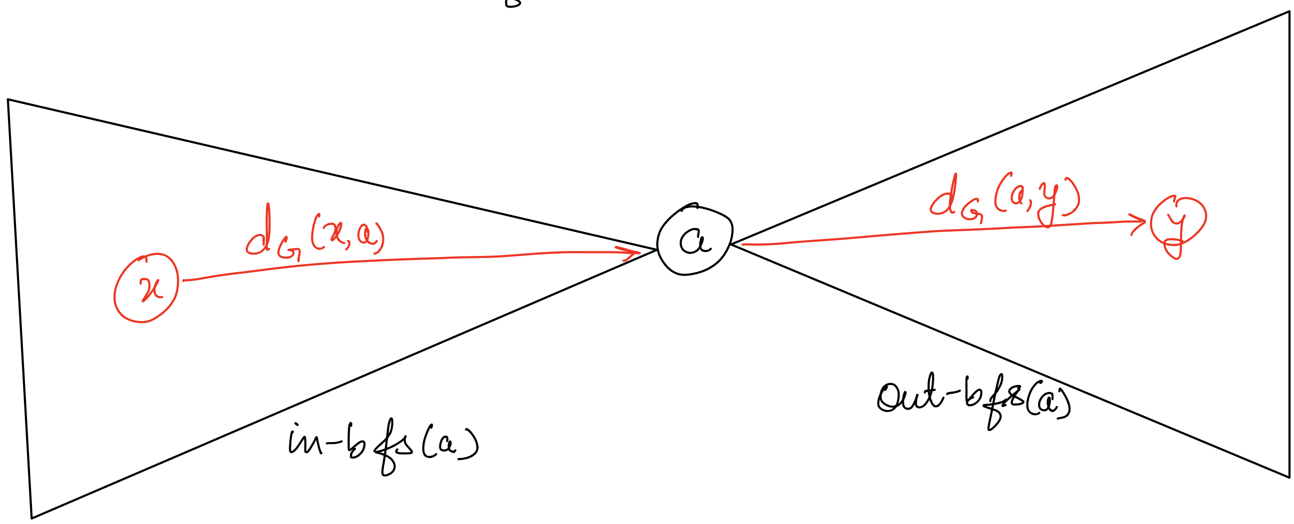
$d_G(a_i, b_j) = 1$

If $H = G \setminus (a_i, b_j)$ then $d_H(a_i, b_j) = \Omega(\alpha)$

so stretch is as high as graph-diameter!

Trivial Approach for Eccentricity Approximation

$a \leftarrow$ centre of G



$$d_G(x,a) + d_G(a,y) \leq ecc_G(x) + ecc_G(a) \leq 2\, ecc_G(a)$$

Ques: Given centre "a" of G, how to find $\hat{ecc}_G(x)$ the estimate of $ecc_G(x)$ lying in range $[ecc_G(x), 2\, ecc_G(x)]$?

Sol^n: Just output $\hat{ecc}_G(x)$ as $d_G(x,a) + ecc_G(a)$

$$\left(= d_G(x,a) + rad(G) \right)$$

H.W. Prove that
$\forall x,\ d_G(x,a) \in [ecc_G(x),\ 2\, ecc_G(x)]$

If we are given a, then the time to

find $(\forall x, \hat{ecc}_G(a))$ is just $O(m+n)$ for unweighted graphs and $O(m \log n)$ for weighted graphs as we need to just compute in-bfs(a) and out-bfs(a).

Drawback of Trivial Approach: The current best algorithm for eccentricity takes $\Omega(\underbrace{\min\{mn, n^{\omega}\}})$ time.

In around this much time we can simply compute all BFS trees / all-pair distances

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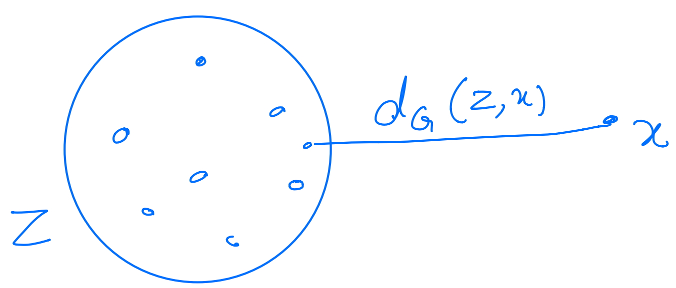
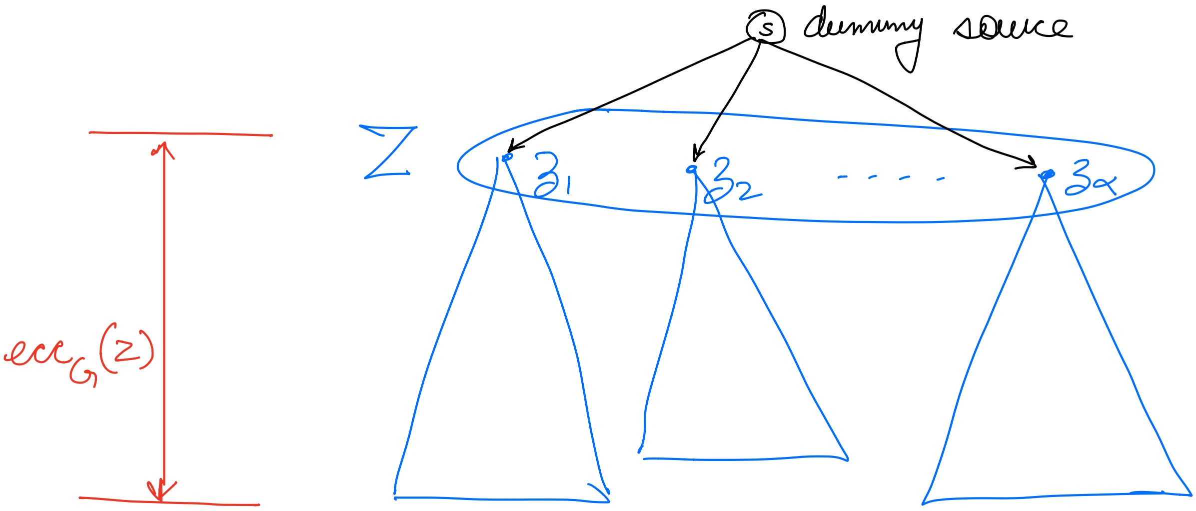
Ideal Goal: Find 2-approximation in $O(m \text{ poly} \log(n))$ time.

<u>Notations</u>

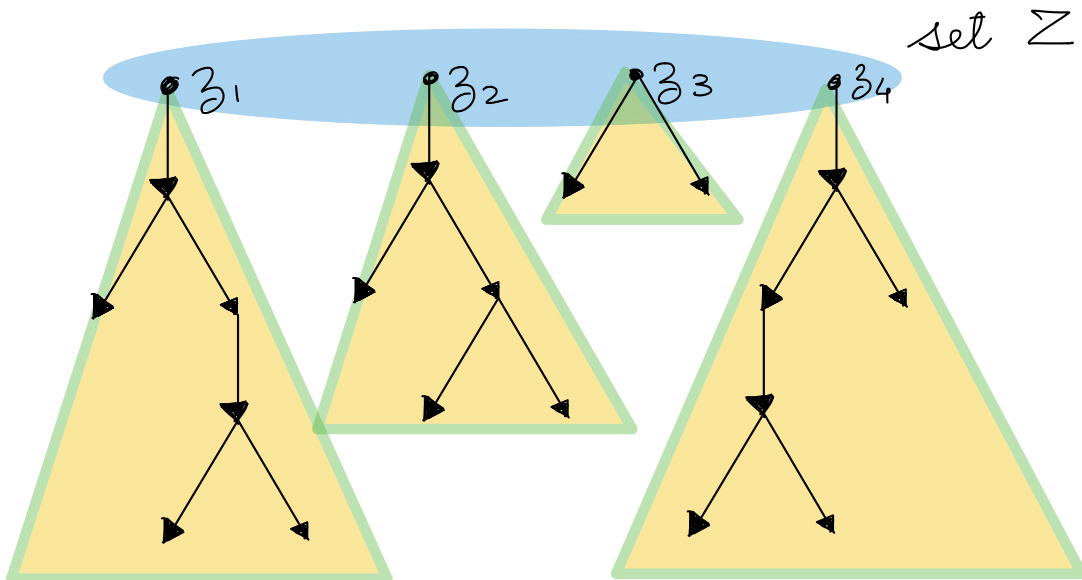
⊛ $d_G(Z, x) = \min_{z \in Z} d(z, x)$



$$(\ast)\quad ecc_G(Z) = \max_{x \in V(G)} d_G(Z, x) \qquad (Z \text{ is set})$$



Example:



$ecc_G(z) = 4$

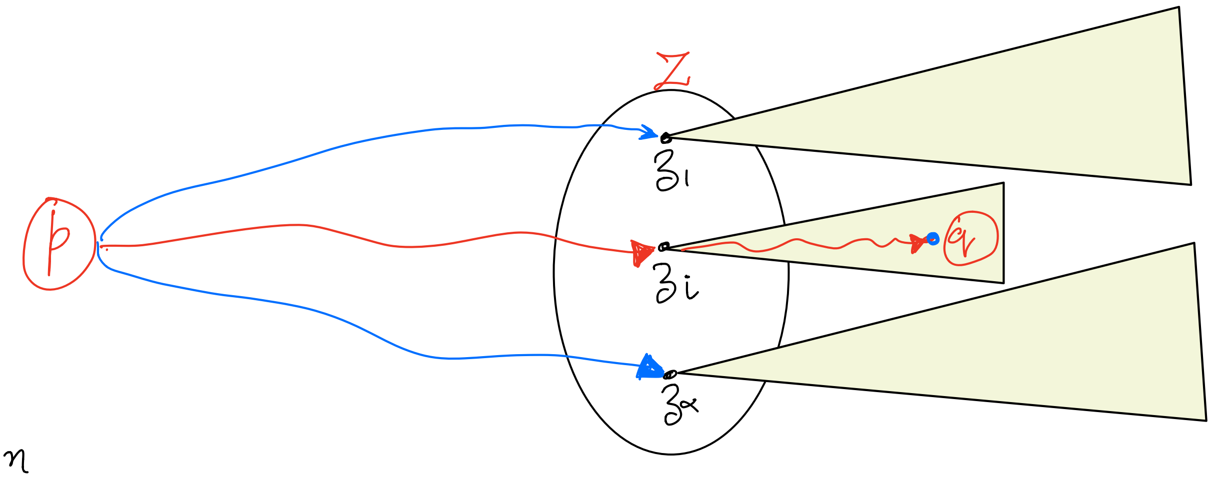
Recall def^n of centre:

A vertex a satisfying $\forall x \in V(G)$, $\boxed{ecc_G(a) \leq ecc_G(x)}$

Def^n (Pseudo centre):

A subset Z of $V(G)$ is said to be pseudo-centre if $\forall\, x \in V(G)$, $\boxed{ecc_G(Z) \leq ecc_G(x)}$

Ques: How to estimate eccentricities using Pseud-centre?



Sol^n

We set $$\boxed{\hat{ecc}_G(p) := \max_{z \in Z} d_G(p, z) + ecc_G(Z)}$$

Note that

(i) $$\hat{ecc}_G(p) := \underbrace{\max_{z \in Z} d_G(p, z)}_{\leq\, ecc_G(p)} + \underbrace{ecc_G(Z)}_{\leq\, ecc_G(p)}$$

$$\leq ecc_G(p) + ecc_G(p)$$

$$\leq 2\, ecc_G(p)$$

(ii) Fix a $q \in V(G)$

Let $z_i \in Z$ satisfy $d_G(Z, q) = d_G(z_i, q)$

Then,
$$d_G(p,q) \le \underbrace{d_G(p, z_i)}_{\downarrow} + \underbrace{d_G(z_i, q)}_{\downarrow}$$
$$\le \max_{z \in Z} d_G(p, z) + ecc_G(Z)$$
$$= \hat{ecc}_G(p)$$

Ques Prove that time to find $\hat{ecc}(x), \forall x$ is $O(m|Z|)$.

Hint : It suffices to compute

(i) In-bfs(z), $\forall z \in Z$

(ii) Out-bfs(Z)

Now the question is how to efficiently compute pseudo-centre Z which is of small size!

Our Z will be of $O(\log^2 n)$ size.

Z will be equal to $R_1 \cup R_2 \cup \cdots \cup R_{\log n}$,

where each R_i is computed using randomization!

Key Property:

Theorem: If W is any given set of vertices, and R is random subset of W of $O(\log n)$ vertices, then we have:

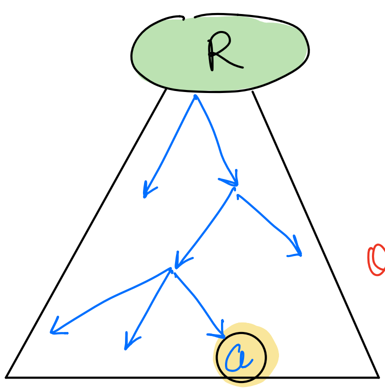
$\exists\ W^* \subseteq W$ of size at least 50% satisfying

R is taking care of 50% vertices

$$\forall x \in W^*, \quad ecc_G(R) \leq ecc_G(x)$$

Proof: Steps to find W^*:

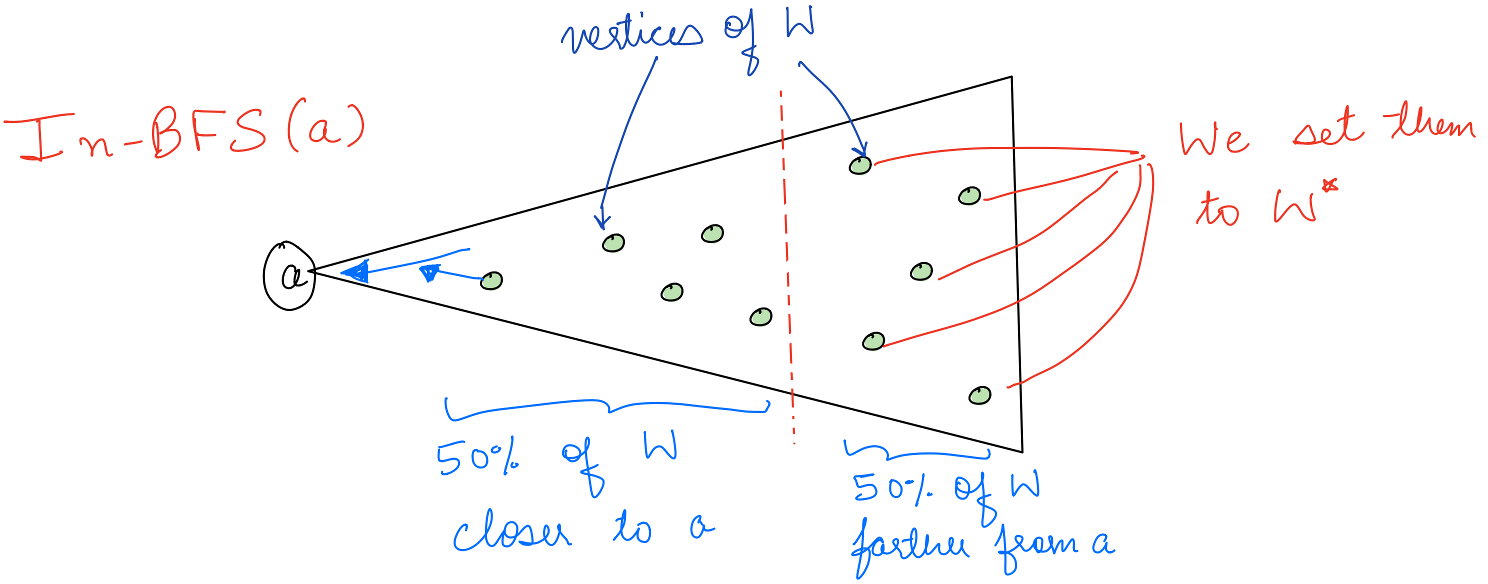
① Find Out-BFS(R), and set $a = \arg\max_v d_G(R, v)$ to be vertex of max-depth



Out-BFS(R)

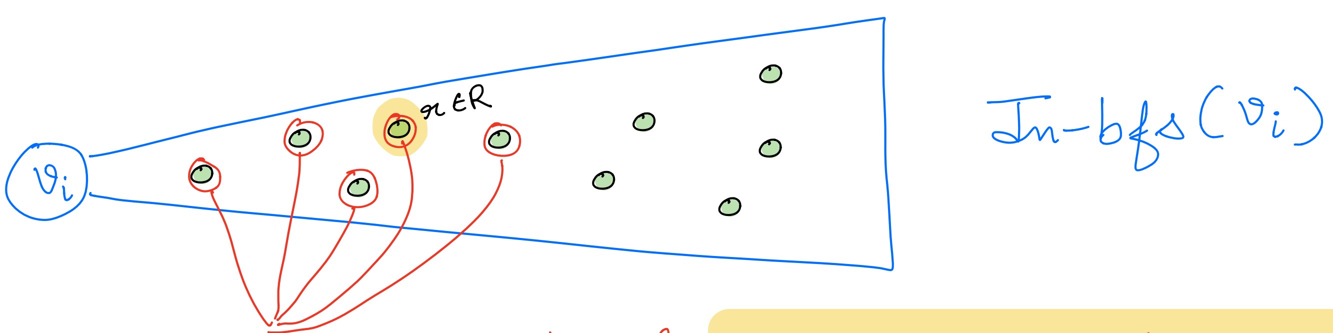
② Compute In-BFS(a) and set

W^* = Those 50% of vertices of W that are farther from a in In-BFS(a).



We now prove correctness of W^* :

Observation 1 : Let $v_1 \ldots v_n$ be vertices of G.



50% vertices of W closer to v_i } has non-empty intersection with R with high probability

Event $\mathcal{E}_i$ = R has non-empty intersection with those 50% of vertices of W that are closer to v_i in In-BFS(v_i)

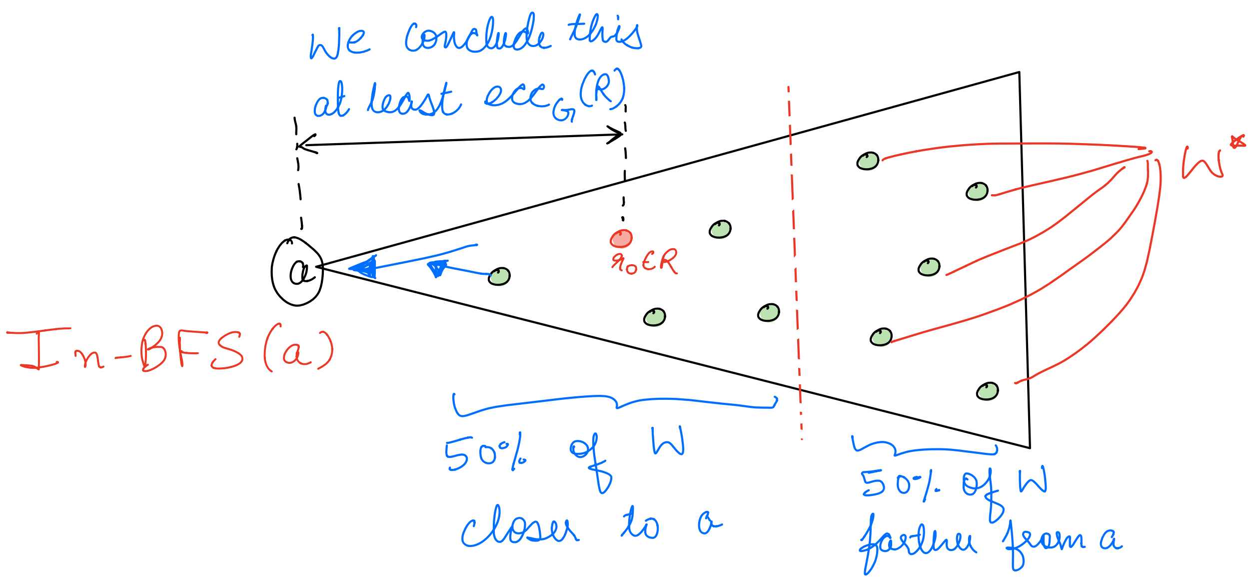
$$\text{Prob}(\mathcal{E}_i^c) \leq \frac{1}{n^{10}}$$

$$\text{Prob}(\mathcal{E}_i^c \text{ holds for at least one } i) \leq \frac{1}{n^9}$$

<u>COROLLARY</u>: R has non empty intersection with those 50% of W that are closer to "a" in In-BFS(a)

Or equivalently $R \cap (W \setminus W^*)$ is non-empty.

<u>Observation 2</u>: Let r_0 be any vertex in $R \cap (W \setminus W^*)$



Note, $ecc_G(R) = d_G(R, a) = \min_{r \in R} d_G(r, a) \leq d_G(r_0, a)$.

<u>Observation 3</u>:

As $r_0 \in W \setminus W^*$, and $W^* = 50\%$ vertices of W farther from "a",

we have $\forall x \in W^*$,

$$d_G(x, a) \geqslant ecc_G(R)$$

This proves the theorem. $\square$

Steps to compute Pseudo-center of G

① Initialize $W_1 = V(G)$.

② For $i = 1$ to $\log_2(n)$:

R_i = Random subset of W_i of $O(\log n)$ size.

a_i = vertex of maximum depth in Out-BFS(R_i).

W_{i+1} = Those 50% of vertices of W_i that are farther from a_i in In-BFS(a_i).

③ Return $Z = (R_1 \cup R_2 \cup \cdots \cup R_{\log_2(n)})$.

Time complexity $= O(m \log n)$

Size of Pseudo-center $= O(\log^2(n))$

Note

(i) $|W_i| = \frac{n}{2^{i-1}}$

(ii) $ecc(R_i) \leq ecc(x), \forall x \in W_i \setminus W_{i+1}$

(by theorem).

(iii) $ecc(z) \leq ecc(R_1), ecc(R_2), \ldots, ecc(R_{\log(n)})$

$ecc(R_1) \leq ecc(x), \quad \forall x \in W_1 \setminus W_2$ — $W_1 \setminus W_2$ taken care by R_1

$ecc(R_2) \leq ecc(x), \quad \forall x \in W_2 \setminus W_3$ — $W_2 \setminus W_3$ taken care by R_2

$\vdots$ $\vdots$

By combining these, we get:

$$ecc(z) \leq ecc(x), \quad \forall x \in V(G).$$

Pseudo-centre of $O(\log^2 n)$ size

So, we get that in $O(m \log n)$ time we can compute pseudo-center of input graph G having size $O(\log^2 n)$.

Next, recall given pesudo-center Z we can compute 2-approximation of eccentricities as:

$$\forall p, \quad \boxed{\hat{ecc}_G(p) := \max_{z \in Z} d_G(p, z) + ecc_G(Z)}$$

Time to evaluate this is $= O(m|Z|)$, as we need to just compute In-BFS(z), $\forall z \in Z$, and OUT-BFS(Z).

Thus, total time to find 2-approximation of all eccentricities is $O(m \log^2(n))$.