(i) 3/2-approximation of graph-diameter

Find $\hat{D}$ satisfying

$$\mathrm{diam}(G) \leq \hat{D} \leq \lceil 1.5\, \mathrm{diam}(G) \rceil$$

Focus on digraphs

(ii) Finding diameter spanner of stretch = 3/2

A sparse subgraph $H = (V, E_H)$ satisfying

$$\mathrm{diam}(H) \leq \lceil k \cdot \mathrm{diam}(G) \rceil$$

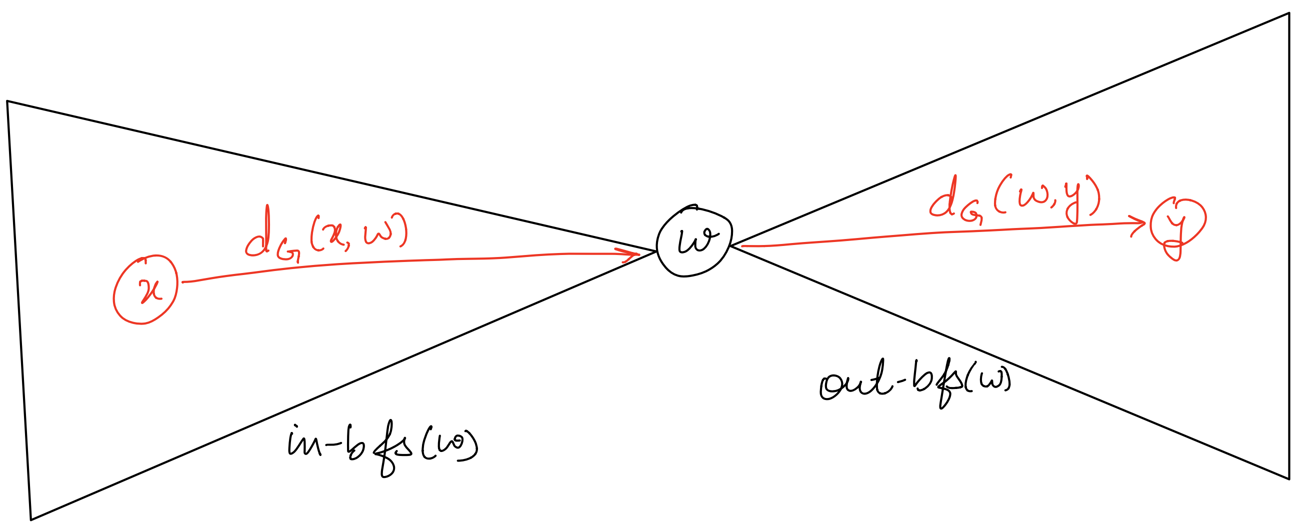
stretch

(*) If G is weighted with edge wts in $[1, W]$, then

$$\mathrm{diam}(H) \leq k \cdot \mathrm{diam}(G) + W$$

$$\mathrm{diam}(G) = \max_{x,y \in V(G)} d_G(x,y)$$

Trivial algorithms for stretch 2



$$d_G(x,w) + d_G(w,y) \le 2\,\mathrm{diam}(G)$$

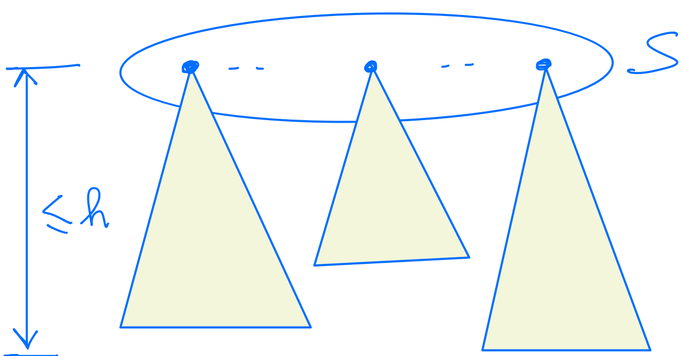
⊛ Trivial 2-approximation:

$$\mathrm{height}(\text{In-bfs}(w)) + \mathrm{height}(\text{Out-bfs}(w))$$

⊛ Trivial 2-diam-spanner:

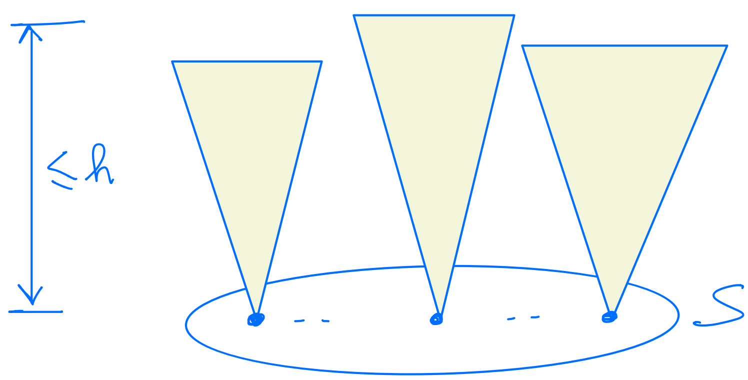
$$H = \left(V,\ E_H = \begin{array}{c}\text{edges-In-BFS}(w)\\ + \\ \text{edges-Out-BFS}(w)\end{array}\right)$$

Def^n: A set "S" is h-out-dominating if $\forall x,\ \mathrm{dist}_G(S, x) \le h$



out-bfs(S)

Def^n: A set "S" is h-In-dominating if $\forall x,\ \mathrm{dist}_G(x, S) \le h$



In-bfs(S)

First Goal: Find in $O(m)$ time a set "S" which is

either $\lceil D/2 \rceil$ - out - dominating

or $\lceil D/2 \rceil$ - in - dominating

Throughout, $D = \text{diam}(G)$.

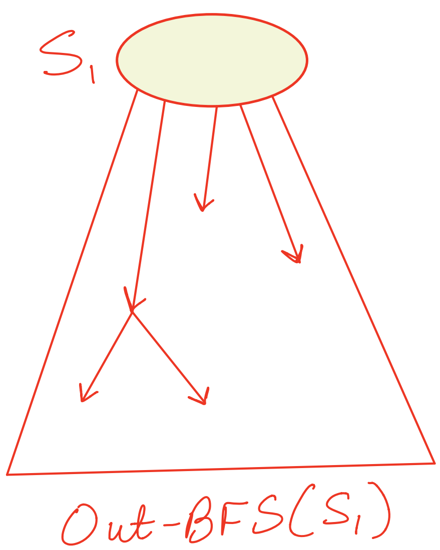
Algorithm:

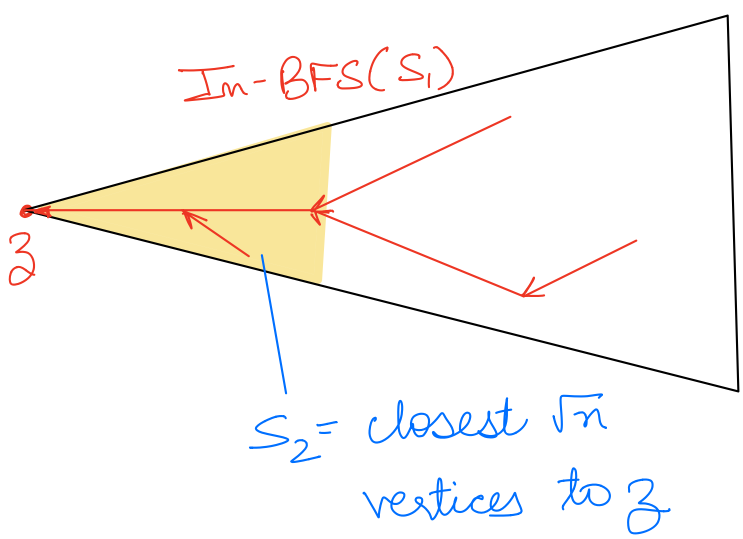
① S_1 = Random subset of $O(\sqrt{n} \log n)$ vertices.

② z = vertex of maximum depth in Out-BFS(S_1).

③ S_2 = $\sqrt{n}$ vertices closest to z in In-BFS(S_1).

④ Output $S = S_1 \cup S_2$.





Lemma: Either S_1 is $\lceil D/2 \rceil$-out-dominating

Or, S_2 is $\lceil D/2 \rceil$-in-dominating.

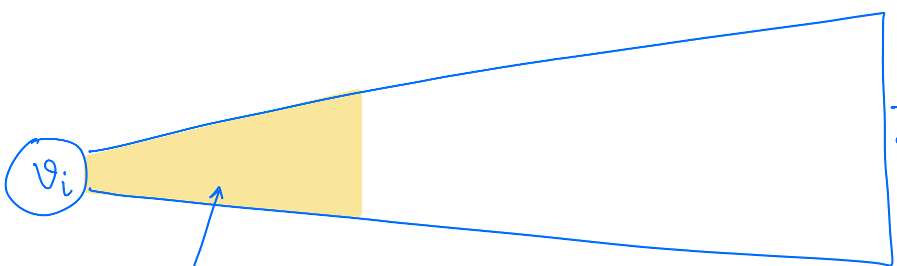
Proof:

If S_1 was $\lceil D/2 \rceil$-Out-dominating then we are done.

So, let us suppose $\text{height}(\text{out-bfs}(S_1)) \not\geq \left\lceil \frac{D}{2} \right\rceil$.

$$\Longrightarrow d_G(S_1, z) \not\geq \left\lceil \frac{D}{2} \right\rceil$$

Observation 1: Let $v_1 \ldots v_n$ be vertices of G.



$\sqrt{n}$ vertices closer to v_i } has non-empty intersection with S_1 with high probability

Event $\mathcal{E}_i = S_1$ has non-empty intersection with $\sqrt{n}$ vertices closest to v_i in In-BFS(v_i).

$$\text{Prob}(\mathcal{E}_i^c) \leq \frac{1}{n^{10}}$$

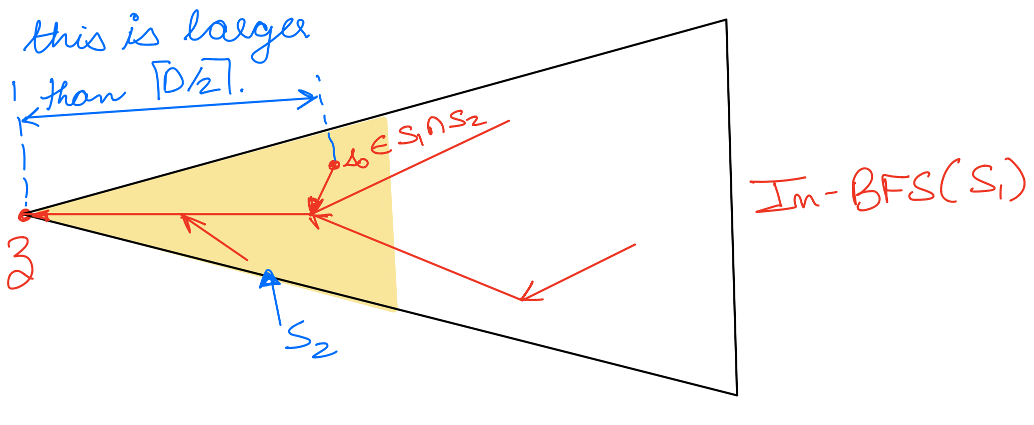
Prob($\mathcal{E}_i^c$ holds for at least one i) $\leq \frac{1}{n^9}$

COROLLARY: S_1 has non empty intersection with $\sqrt{n}$ vertices closest to "z" in tree In-BFS(z).

Or, equivalently $S_1 \cap S_2$ is non-empty.

Observation 2: Let $s_0 \in S_1 \cap S_2$, then

$$d_G(s_0, z) \geq d_G(S, z) \not\geq \lceil D/2 \rceil.$$



Observation 3:

$$\text{depth}(\text{In-BFS}(S_2)) \leq \lceil D - \lceil D/2 \rceil \rceil \leq \lceil D/2 \rceil.$$

Thus, for all x, $d_G(S_2, x) \leq \lceil D/2 \rceil$.

So, if S_1 is not $\lceil \frac{D}{2} \rceil$ - out - dominating, then with high probability S_2 is $\lceil \frac{D}{2} \rceil$ - in - dominating.

$\Rightarrow S = S_1 \cup S_2$ must be either

$\lceil \frac{D}{2} \rceil$ - out - dominatig, or

$\lceil \frac{D}{2} \rceil$ - in - dominating.

Ques. If we are given an S which is

(i) Either $\lceil D/2 \rceil$ - out - dominating,

(ii) Or, $\lceil D/2 \rceil$ - in - dominating.

Then how to distinguish b/w them in $O(m)$ time without knowing the value of D.

Sol^n: compute In-BFS(S) and out-bfs(S)

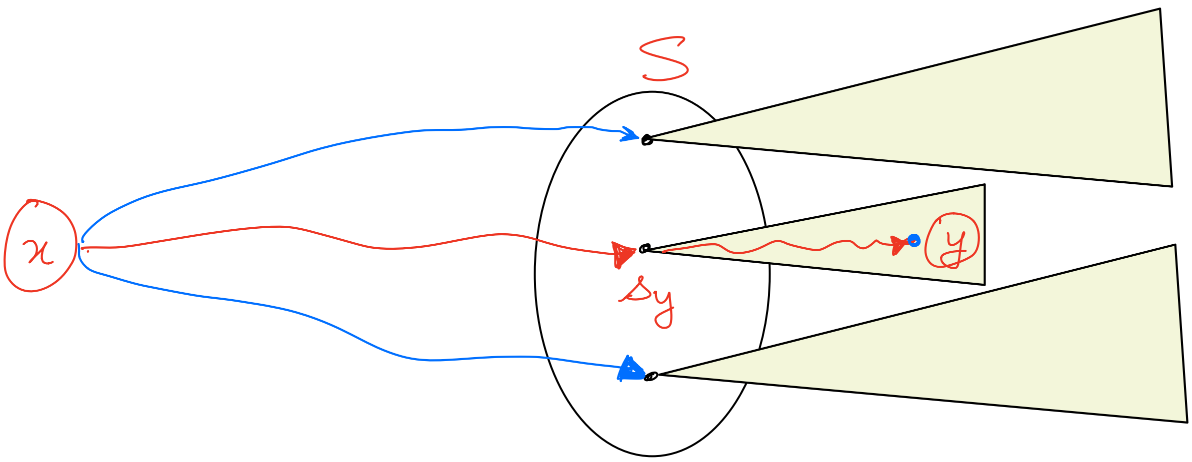
If height(out-BFS(S)) is smaller $\Rightarrow$ (i) holds

If height(In-BFS(S)) is smaller $\Rightarrow$ (ii) holds

Diameter Approximation using set "S"

We will consider the case S is $\lceil D/2 \rceil$-out-dominating.
(The algorithm & analysis of case S is $\lceil D/2 \rceil$-in-dom. will be symmetric)

Sol^n



We set $\hat{D} := \max\limits_{(x,s) \in V(G) \times S} d_G(x, s) + ecc_G(S)$

(i) Claim: $diam(G) \leq \hat{D}$

To prove the claim consider a pair $(x,y) \in V \times V$.

Let $s_y \in S$ be such that $d_G(s_y, y) = d_G(S, y)$.

Now observe,

$$d_G(x, s_y) \leq \max_{(x,s) \in V(G) \times S} d_G(x, s)$$

$$d_G(s_y, y) = d_G(S, y) \leq ecc_G(S)$$

So, $d_G(x,y) \leq d_G(x, s_y) + d_G(s_y, y) \leq \hat{D}$.

Thus $\mathrm{diam}(G) = \max_{x,y \in V} d_G(x,y) \leq \hat{D}$.

(ii) Claim: $\hat{D} \leq \lceil 1.5\, \mathrm{diam}(G) \rceil$

To prove the claim observe that

(*) $\max_{(x,s) \in V(G) \times S} d_G(x, s) \leq \mathrm{diam}(G)$

(*) $ecc_G(S) \leq \left\lceil \frac{\mathrm{diam}(G)}{2} \right\rceil$ as S is $\lceil \frac{D}{2} \rceil$-out-dom

On combining, we get $\hat{D} \leq \lceil 1.5\, \mathrm{diam}(G) \rceil$.

Time taken to find $\hat{D}$ $= O(m|S|) +$ Time to find set S $= O(m\sqrt{n} \log n)$.

Computing diameter-spanner using set "S"

We can set

$$H = \left(V,\ E_H = \bigcup_{s \in S} \begin{array}{l} \text{edges of In-BFS}(s), \ \& \\ \text{edges of Out-BFS}(s) \end{array} \right)$$

We prove correctness of the case S is $\lceil D/2 \rceil$-out-dominating.

(The algorithm & analysis of case S is $\lceil D/2 \rceil$-in-dom. will be symmetric)

To prove correctness, consider a pair $(x,y) \in V \times V$.

Let $s_y \in S$ be such that $d_G(s_y, y) = d_G(S, y)$.

Now observe,

$$d_H(x,y) \le d_H(x, s_y) + d_H(s_y, y)$$

$$= d_G(x, s_y) + d_G(s_y, y)$$

$$= d_G(x, s_y) + d_G(S, y)$$

$$\leq \operatorname{diam}(G) + ecc_G(S)$$

$$\leq \lceil 1.5 \operatorname{diam}(G) \rceil \quad \left(\text{bcoz } S \text{ is } \lceil D/2 \rceil\text{-out-dom} \right)$$

Thus,

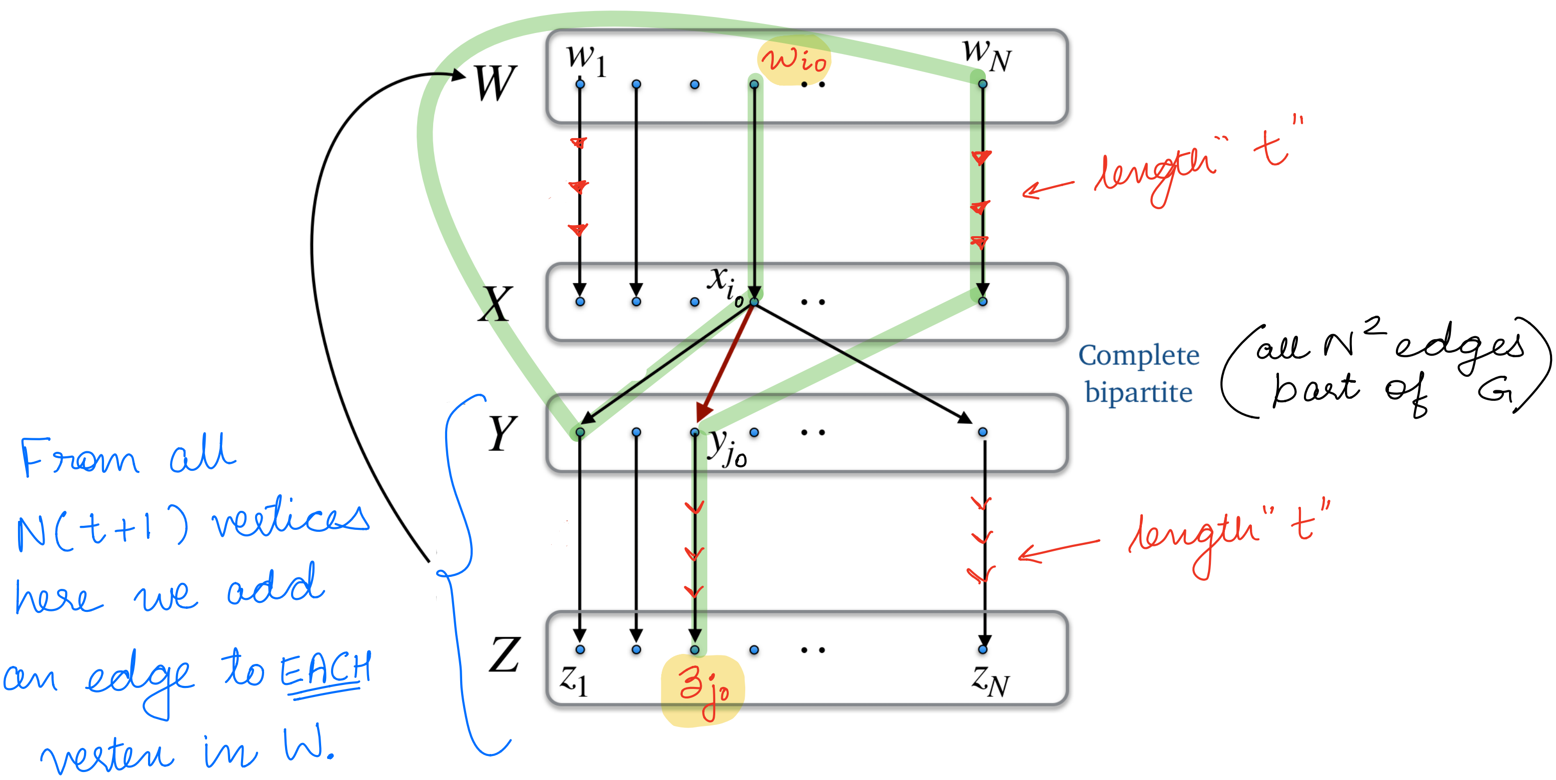
$$\operatorname{diam}(H) = \max_{x,y} d_H(x,y) \leq \lceil 1.5 \operatorname{diam}(G) \rceil.$$

- No. of edges in H $= O(n\,|S|)$ $= O(n\sqrt{n}\log(n))$
- Time to compute H $= O(n\,|S|)$ $= O(n\sqrt{n}\log(n))$

Lower Bound

Theorem: For any n, there exists $\Theta(n)$ vertex graph G whose diameter-spanner of stretch less than 1.5 requires $\Omega(n^2)$ edges.

Proof: Consider following construction of G



$$|V(G)| = 2 \cdot N \cdot (t+1) = n \text{ (say)}$$

$\operatorname{diam}(G) = 2(t+1)$ as $d_G(z_i, z_j) = 2(t+1)$, for $i \neq j$.

If H is sparser than G, then it cannot contain all N^2 edges from $(X) \times (Y)$.

Let $H = G \setminus (x_{i_0}, y_{j_0})$ for some $i_0, j_0 \in [1, N]$.

Then,

$\mathrm{diam}(H) = 3(t+1)$ bcoz $d_H(w_{i_0}, z_{j_0}) = 3(t+1)$.

For t to be some constant, $N = \Theta(n)$.

Thus, if want H to have stretch strictly less than 1.5, then H must contain $\Omega(n^2)$ edges.

$\square$

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1 A Simple Construction of 1.5-Diameter Spanner

Construction: Let

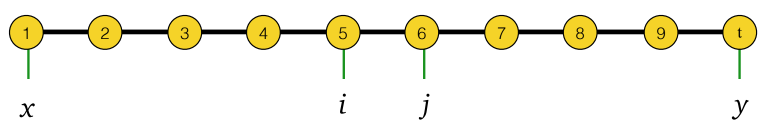
$$X_{out} = \{v \in V \mid \textsc{out-deg}(v) > \sqrt{n}\}, \text{ and}$$
$$X_{in} = \{v \in V \mid \textsc{in-deg}(v) > \sqrt{n}\}$$

be two subset of vertices of G, containing respectively the vertices having high out-degree and high in-degree in G. The algorithm for 1.5-diameter-spanner computation is given below.

Initialize $H = (V, E_H = \emptyset)$ to be an empty graph;
foreach $v \notin X_{out}$ **do** add all outgoing edges of v to E_H;
foreach $v \notin X_{in}$ **do** add all incoming edges of v to E_H;
Find a random hitting set S of $O(\sqrt{n}\log n)$ vertices that contains at least one out-neighbor of each vertex in X_{out}, and at least one in-neighbor of each vertex in X_{in};
foreach $s \in S$ **do**
    Add to E_H union of the edges in trees $\textsc{in-bfs}(s)$ and $\textsc{out-bfs}(s)$;
Return $H = (V, E_H)$;

The time for computing H is $O(m\sqrt{n}\log n + n)$, and $|E_H|$ is $O(n\sqrt{n}\log n)$.

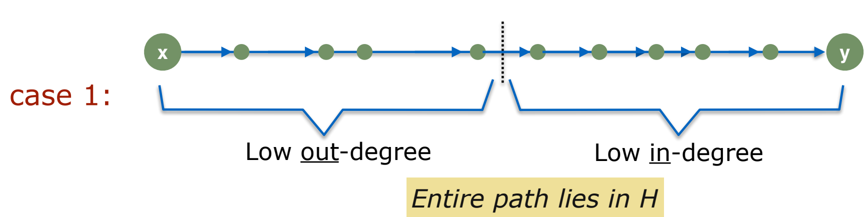
Correctness: Consider two distinct vertices $x, y \in V$. Let $P = (x = a_1, \ldots, a_t = y)$ be the shortest-path from x to y in G. Let $i \in [1, t]$ be the largest index such that $d_G(x, a_i) < d_G(x, y)/2$, and similarly, let $j \in [1, t]$ be the smallest index such that $d_G(a_j, y) < d_G(x, y)/2$.



Claim 1. *Either $i = j - 1$ or $i = j - 2$.*

To prove the correctness we consider three different cases.

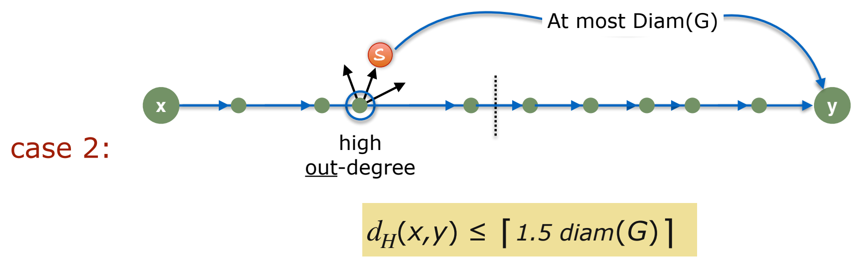
Case 1. If all vertices in $P[x, a_i]$ have low out-degree (i.e., at most $\sqrt{n}$), and all vertices in $P[a_j, y]$ have low in-degree (i.e., at most $\sqrt{n}$), then P is trivially contained in H.



Case 2. We now analyse the scenario when $P[x, a_i]$ contains a vertex, say w, of out-degree larger than $\sqrt{n}$. Let s be the out-neighbour of w present in the hitting-set S. If G is unweighted, then it can be verified that

$$d_H(x,s) = d_G(x,s) \leq_{(why?)} \lceil d_G(x,y)/2 \rceil \leq \lceil diam(G)/2 \rceil.$$

Since $d_H(s,y) = d_G(s,y) \leq diam(G)$, from the triangle inequality (on vertices x, y, s in H), we obtain that $d_H(x,y)$ is at most $\lceil 1.5\, diam(G) \rceil$.



Case 3. The analysis of the scenario when $P[a_j, y]$ contains a vertex of in-degree larger than $\sqrt{n}$ will follow similar to that of Case 2.

Remark for weighted graphs If G has edge-weights from the range $[1, M]$, we substitute every execution of a BFS with Dijkstra's algorithm for finding a shortest-path tree. Then it can be verified that $d_H(x,s) = d_G(x,s) \leq d_G(x,y)/2 + M \leq diam(G)/2 + M$, and so $d_H(x,y)$ is at most $1.5\, diam(G) + M$.

From the discussion above, we obtain the following theorem.

Theorem 1. *For any unweighted directed graph $G = (V, E)$ with n vertices and m edges, there exists a (1.5)-diameter spanner $H = (V, E_H \subseteq E)$ with at most $O(n^{3/2} \log n)$ edges that is computable in $O(m\sqrt{n} \log n + n)$ time.*

Further, if G is edge-weighted with weights from the range $[1, M]$, then H satisfies

$$diam(H) \leq (1.5) diam(G) + M.$$