Fault Tolerant Spanner.

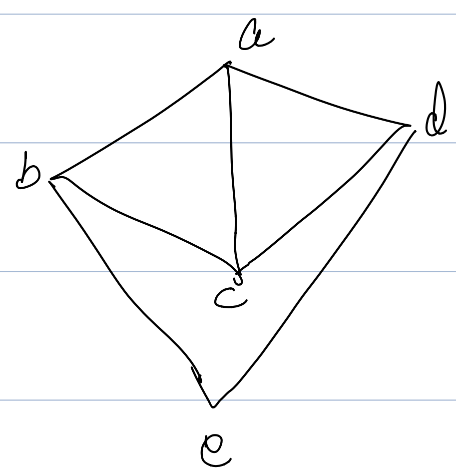
* Fault Tolerant Network Prone to r failures.

  vertices edges both

Given: original graph G,
parameter "r".

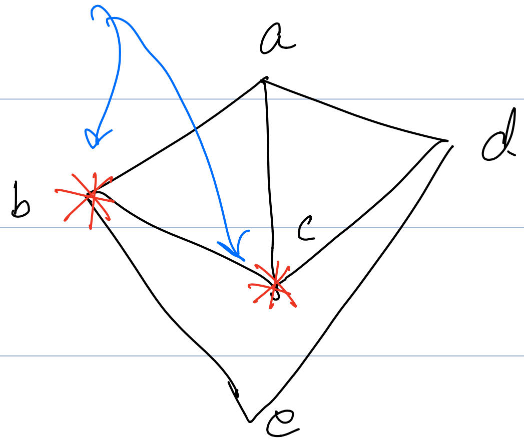
In Future: New graph $\tilde{G}$ has at most r failures at any given time. (E.g. LAN or road network).

EXAMPLE ($r=2$):

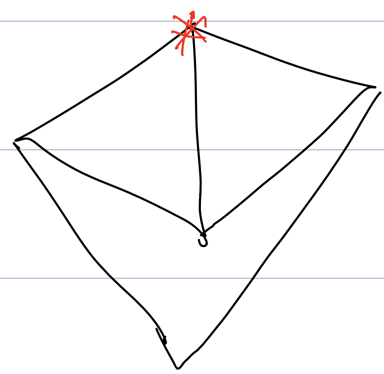


G: original graph

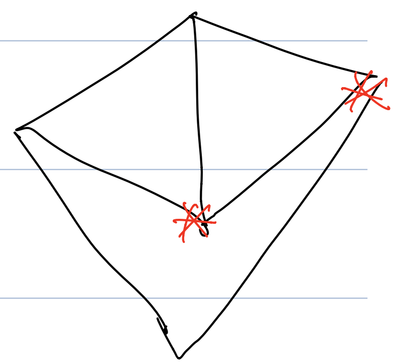
$\tilde{G}$ has 2 node failures



time $t=1$
$F=\{b,c\}$



time $t=2$
$F=\{a\}$

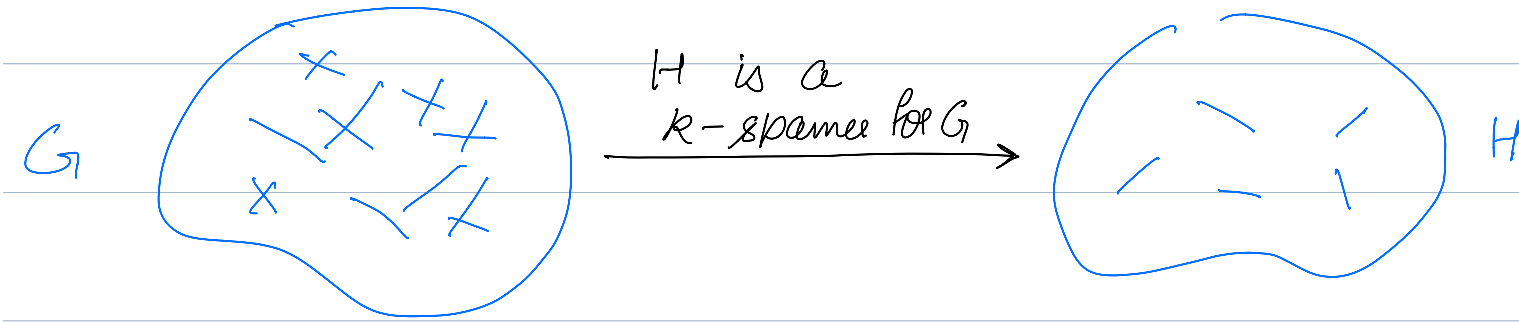


time $t=3$
$F=\{c,d\}$

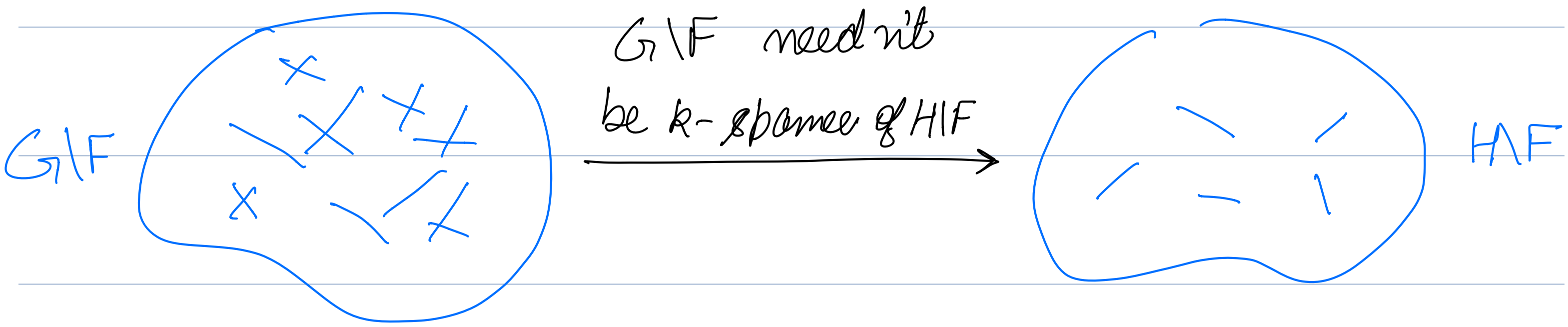
Recall k-Spanners:

Sparse subgraph $H = (V, E_H \subseteq E)$ of $G = (V, E)$ satisfying

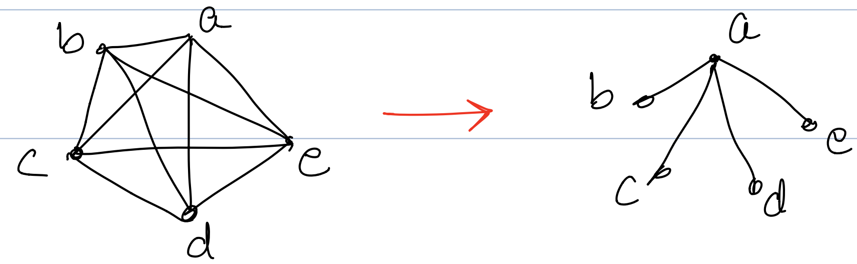
$$\forall x, y \quad dist_H(x,y) \leq k \cdot dist_G(x,y)$$



In scenario of $(\leq r)$ failures, we have:



Eg: $G = K_5$ $\qquad$ H : spanner with stretch $= 2$



$F = \{a\}$: $\quad G \setminus \{a\}$ is complete graph $\longrightarrow$ $H \setminus \{a\} \equiv$ empty

$\therefore$ stretch $= \infty$

Our Goal: Compute a sparse subgraph H of G s.t. $\forall\ F \subseteq V(G)$ of size $\leq r$.

$H \setminus F$ is a k-spanner of $G \setminus F$.

That is,

$\forall x, y \in V \times V$
$\forall\ F \subseteq V$ of size $\leq r$ $\Big\}$ $d_{H \setminus F}(x,y) \leq k \cdot d_{G \setminus F}(x,y)$

⊛ $k \leftarrow$ stretch

⊛ $r \leftarrow$ Max number of vertex failures

$H \leftarrow$ "r - FT - k - spanner"

Max number of failures

fault tolerant

stretch

Notation : $F \equiv$ Set of failures in G

Result (Vertex-failures)

If for every graph G with n vertices we can compute k-spanner of $f_k(n)$ size $\rightarrow$ Then, we can compute for n-vertex graph a r-vertex-FT-k-spanner H of $O(r^3 \log n \cdot f_k(n))$ size.

Def^n: An edge (x, y) is said to be handled in H, if $\forall\ F \subseteq V(G) \setminus \{x, y\}$ of size at most r, we have: $dist_{H \setminus F}(x, y) \leq k$.

(Note: $dist_{G \setminus F}(x, y) = 1$, so stretch in $H \setminus F$ w.r.t. $G \setminus F$ should be at most "k".)

Lemma 1: If each $(x, y) \in E(G)$ is handled in H, then H is r-FT-k-spanner.

Proof: H.W.

⊛ A-Simple Algorithm to compute r-FT-k-spanner.

1. $H = (V, \emptyset)$
2. $\forall F \subseteq V$ of size r:
   - $H_F \leftarrow k$-spanner of $G \setminus F$
   - Add edges of H_F to H

This is a bad algo. Why?

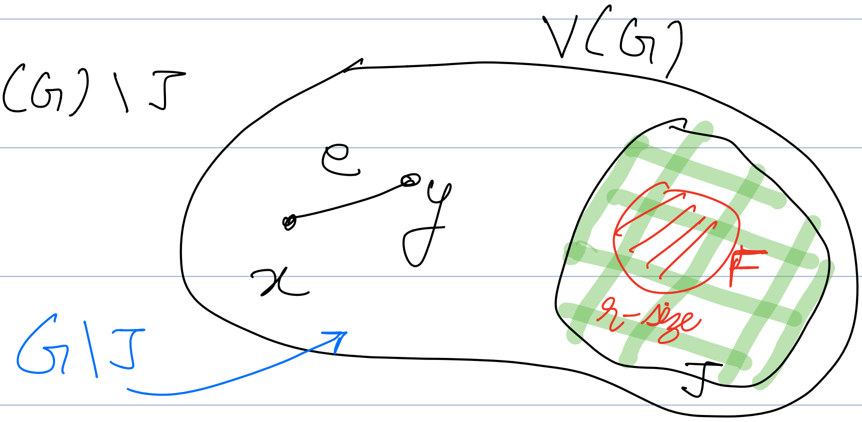
Because step-2 runs ${}^{n}C_{r}$ times.

This will NOT result in sparse H.

Def^n: A set "J" is said to separates e and F if

(i) $x, y \in V(G) \setminus J$

(ii) $F \subseteq J$



Main Idea:

We find sets $J_1 \; J_2 \ldots J_\alpha$ s.t.

$\forall$ edges $e = (x, y)$, &

$\forall$ $F \subseteq V \setminus \{x, y\}$ failing vertex-set of r size

there is a J_i separating e and F.

(That is, $x, y \in V(G) \setminus J_i$
$F \subseteq J_i$).

(*) In such scenario we can include spanner of $V(G) \setminus J_i$ in main graph "H".

⊛ An Improved Algorithm to compute r-FT-k-spanner.

① $H = (V, \phi)$

② For $i = 1$ to $\alpha = 16 r^3 \log(n)$

- Compute set J_i s.t. each vertex is included in J_i with prob. $= \left(1 - \frac{1}{r}\right)$
- $H_i \leftarrow$ k-spanner of $G | J_i$
- Add edges of H_i to H.

If $f_k(n)$ best possible size of k-spanner on n vertices $\xrightarrow{\text{Then}}$ Size of H is $\alpha \cdot f_k(n)$

To prove correctness we define 3 events:

① $A(i, e=(x,y), F) \equiv$ Event that J_i separates e and F

② $A(e=(x,y), F) \equiv$ Event that e & F are separated by some J_i

③ $B \equiv$ Event that $\forall\, e=(x,y)$ & $\forall\, F \subseteq J$ of r size not containing x, y, we have e is separated from F by some J_i.

Our Goal: To show that event B holds with high probability

H.W.

$$\boxed{\frac{1}{2} \leq \left(1-\frac{1}{r}\right)^{r} \leq \frac{1}{e}}$$

① $\text{Prob}(A(i, e=(x,y), F)) = \underbrace{(1-p)}_{x \notin J_i} \underbrace{(1-p)}_{y \notin J_i} \underbrace{p^{r}}_{F \text{ in } J_i}$

$p = 1 - \frac{1}{r}$

$$= \frac{1}{r} \cdot \frac{1}{r} \cdot \left(1 - \frac{1}{r}\right)^{r} \geq \frac{1}{4r^2}$$

$\boxed{\text{Prob}(\neg A(i,e,F)) \leq \left(1 - \frac{1}{4r^2}\right) \text{ is NOT SMALL} \quad \text{But, we decrease prob of BAD EVENT by repeating } \alpha \text{ times}}$

② $A(e,F)$ holds $\equiv \exists\, i$ s.t. $A(i,e,F)$ holds

$\neg A(e,F)$ holds $\equiv \forall\, i \quad \neg A(i,e,F)$ holds

Note: $J_1\ J_2 \ldots J_\alpha$ are computed independently

So, $\text{Prob}(\neg A(e,F)) = \prod_{i=1}^{\alpha} \text{Prob}(\neg A(i,e,F))$

$$\leq \left(1 - \frac{1}{4r^2}\right)^{\alpha} = \left(1 - \frac{1}{4r^2}\right)^{16 r^3 \log n}$$

$$= \left(1 - \frac{1}{4r^2}\right)^{4r^2 * 4r \log n} \leq e^{-4r \log n} = \frac{1}{n^{4r}}$$

(3) B holds $\equiv \forall e, F$ $A(e,F)$ holds

$\neg B$ holds $\equiv \exists e, F$ s.t. $\neg A(e,F)$ holds

By union bound,

$$\text{Prob}(\neg B) \leq \sum_{\substack{e \in E(G) \\ F \subseteq V(G) \text{ of size } r}} \neg A(e,F)$$

Max count of edges

$$\leq \underbrace{{}^{n}C_2}_{} \cdot \underbrace{{}^{n}C_r}_{} \cdot \frac{1}{n^{4r}}$$

Choices for set F of V(G) of r size

$$\leq \frac{n^{r+2}}{n^{4r}} \leq \frac{1}{n^{3r}} \leq \frac{1}{n^3}$$

As probability $(\neg B) \leq \frac{1}{n^3}$ with probability $\left(1-\frac{1}{n^3}\right)$ we have that each $e \in E(G)$ and each $F \subseteq V$ of r size not containing x, y are separated by some J_i.

The correctness of algo is evident from following lemma.

Lemma : Let $J_1\ J_2 \ldots J_\alpha$ be subsets of $V(G)$ s.t. $\forall\, e = (x, y)$, $\forall\, F \subseteq V(G) \setminus \{x, y\}$ of size r, e and F are separated by at least one J_i.

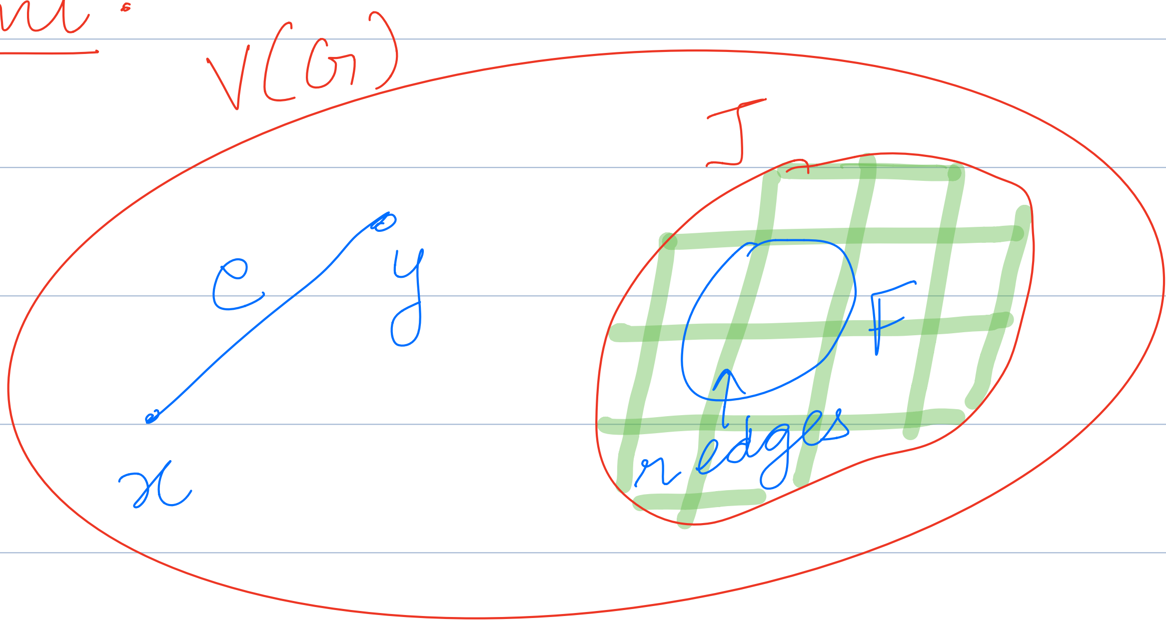
Also let H_i be k-spanner of $G \setminus J_i$.

Then $H = \left(V, \bigcup\limits_{1 \leq i \leq \alpha} E(H_i)\right)$ is "r-FT-k-spanner" of G.

Proof : H.W.

Question: How will you compute r-edge-FT-k-spanner?

Hint:



You need to separate e from $F \subseteq E(G)$ of size r

So, J_i should be random set of $E(G)$.
$i = 1$ to α.

* Each edge is included in J_i with probability p

Ques. What is optimal values for p & α?

$$\text{Prob}(e, F \text{ are separated by } J_i) = (1-p)\cdot p^r$$

Ques If you set $p = \frac{1}{r}$ then what should be α?

Ques. What will be size of final r-Edge-FT-k-spanner in terms of n and r?

— X — X —