Def^n: r-edge-FT-BFS

Given : graph $G=(V,E)$ ← unweighted

Fault Parameter "r"

source set "S" of size z.

Can be directed / undirected

Find : $H=(V, E_H \subseteq E)$ a subgraph of $G=(V,E)$ s.t.

$\forall$ failing-set $F \subseteq E(G)$ of size "r"

$\forall\ (s,v) \in S \times V$, $dist_{G \setminus F}(s,v) = dist_{H \setminus F}(s,v)$.

Preserve distances for pairs in $S \times V$ as long as failures $\leq r$.

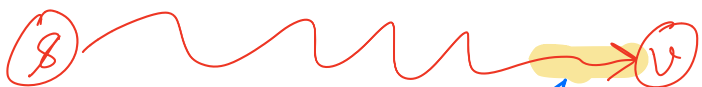
$\Rightarrow$ That is, $H \setminus F$ contains BFS tree of $G \setminus F$, $\forall\ s \in S$.

Main Idea: Focus on one vertex "v"

Find a set $\mathcal{E}_v^r(S,G)$ of edges of v s.t.

If $dist_{G\setminus F}(s,v) \neq \infty$

Then $\longrightarrow$ $\exists$ s-v path in $G\setminus F$ with last edge in $\mathcal{E}_v^r(S,G)$



last edge is in $\mathcal{E}_v^r(S,G)$

Theorem 1: Let $E_0 \subseteq E(G)$ be s.t. $\forall w \in V(G)$ and $\forall$ F of size r with $dist_{G\setminus F}(s,w) \neq \emptyset$:

$\exists$ a s-w shortest-path in $G\setminus F$ with last edge in E_0.

Then, $G_0 = (V, E_0)$ is a r-FT-BFS.

Proof: H.W.

(*) we will find $\forall v \in V(G)$, a set $\mathcal{E}^{r}_{v}(S,G)$ s.t. ← failure count

"$\bigcup_{v \in V(G)} \mathcal{E}^{r}_{v}(S,G)$" satisfy condⁿ of Thm 1.

How to find set: $\mathcal{E}^{r}_{v}(S,G)$, for a given "v" ?

(*) In order to find r-FT-BFS(S, G) we will use (r-1)-FT-BFS(S_0, G_0) for some different S_0, G_0.

i.e.

We are trying to reduce problem of r failure to (r-1) failures.

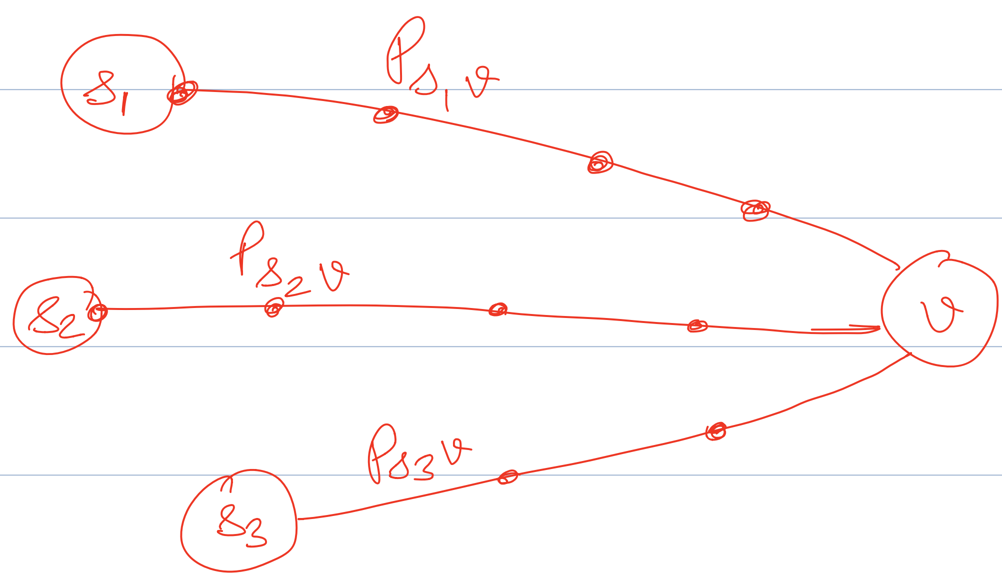
Notation

$\Pi_{G \setminus F}(s,v)$ ← s-v shortest path in $G \setminus F$.

$P_{s,v}$ ← s-v shortest path in G.

Partial Tree Structure of $P_{s,v}$, for $s \in S$.

Eg:



$S = \{s_1, s_2, s_3\}$ $\quad \left| \bigcup_{s \in S} EDGE(P_{s,v}) \right| = 11$

* "G_0" will be $G \setminus \bigcup_{s \in S} EDGES(P_{s,v})$

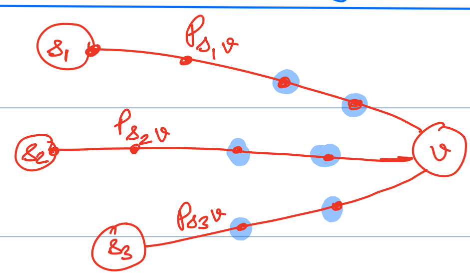
* What will be S_0?

* How to reduce r failures to $r-1$?

"Finding S_0"

$t \leftarrow$ a parameter

Eg: Last 2-vertices



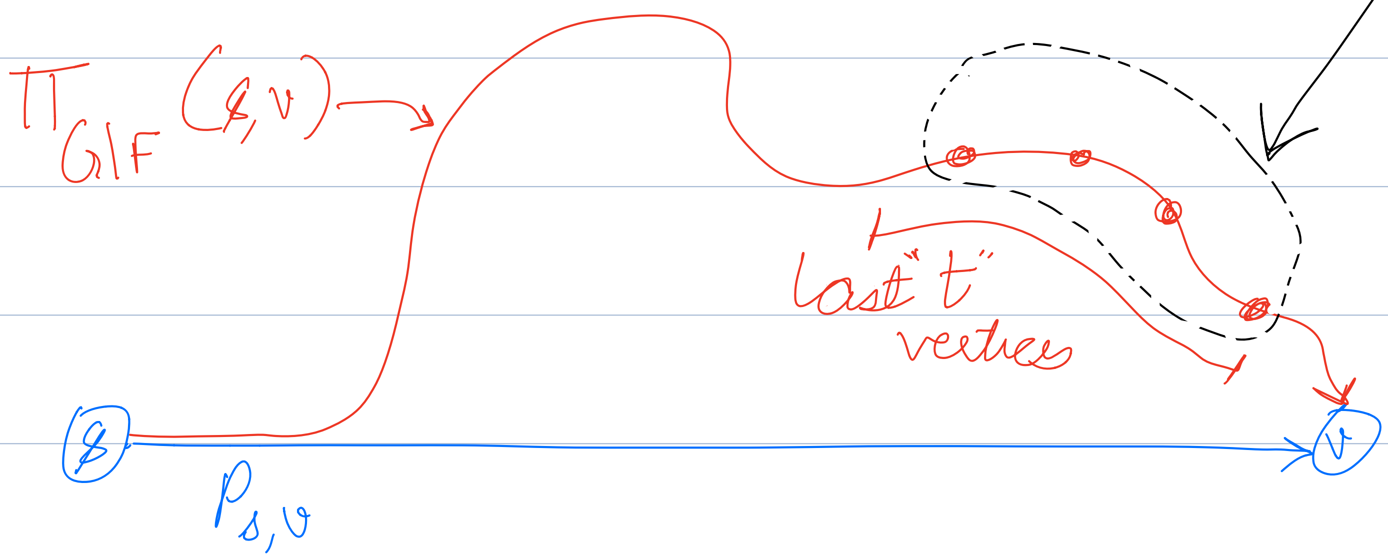
S_0

L $\rightarrow$ $\bigcup_{s \in S}$ Last t vertices in $P_{s,v}$

R $\rightarrow$ A random hitting set that hits each set in collection:

$$\mathcal{P} = \left\{ \{\text{Last-}t\text{-vertices of } \pi_{G\setminus F}(s,t)\} \;\middle|\; \begin{array}{l} s \in S \\ F \subseteq E \text{ of size } r \\ d_{G\setminus F}(s,v) \geq t \end{array} \right\}$$

If $t=4$, then this set of last t vertices is one element of $\mathcal{S}$



(*) Exercise: Hitting set R will have size $O\left(\frac{n}{t}\, r \log n\right)$.

Hint: $|\mathcal{S}| \leq |S| \cdot {}^{n^2}C_r < n^{1+2r} \leq n^{3r}$

$$\mathcal{E}^{r}_{v}(S, G) = \overset{\text{LUR}}{\overset{\parallel}{\mathcal{E}^{r-1}_{v}(S_0, G_0)}} + \bigcup_{s \in S} \text{last-edge}(P_{s,v})$$

CORRECTNESS

Consider a $s \in S$ and F of r size

Possible Cases

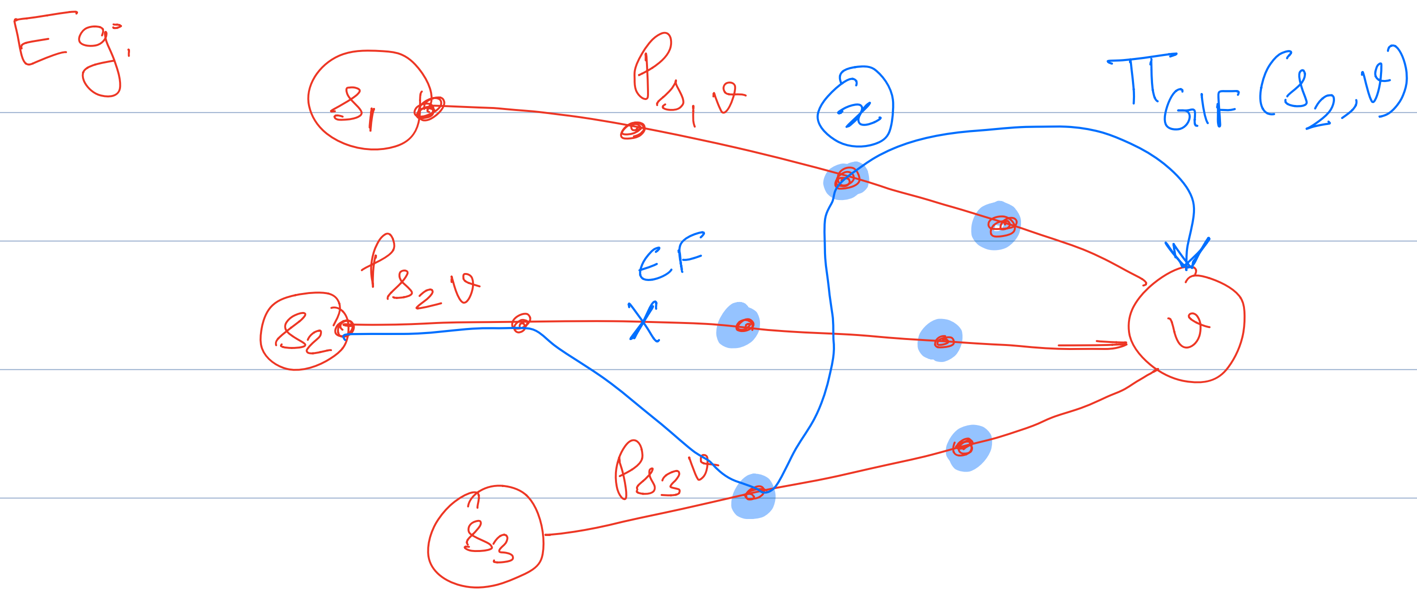
① There is s-v shortest path in $G \setminus F$ with last-edge in $\left(\bigcup_{s \in S} \text{last-edge}(P_{s,v})\right)$

(we are already done here).

② There is no s-v shortest path in $G \setminus F$ with last-edge in $\left(\bigcup_{s \in S} \text{last-edge}(P_{s,v})\right)$

Let x = last vertex in $\pi_{G \setminus F}(s,t)$ lying in $\boxed{\bigcup_{s \in S} \text{vertices}(P_{s,v})}$

Eg:



Illustation of last t vertices for $t = 2$

This implies $\Pi_{G \setminus F}(s,v)[x, v]$ lie in G_0

bcoz it can't contain edges of $\left(\bigcup_{s \in S} P_{s,v} \right)$

Let $\boxed{F_0}$ be set of those edges in F that lie in G_0. $\quad (F_0 = F \cap E(G_0))$

Note: $|F_0| \leq |F| - 1$ bcoz at least one failure lie in Edges$(P_{s,v})$!

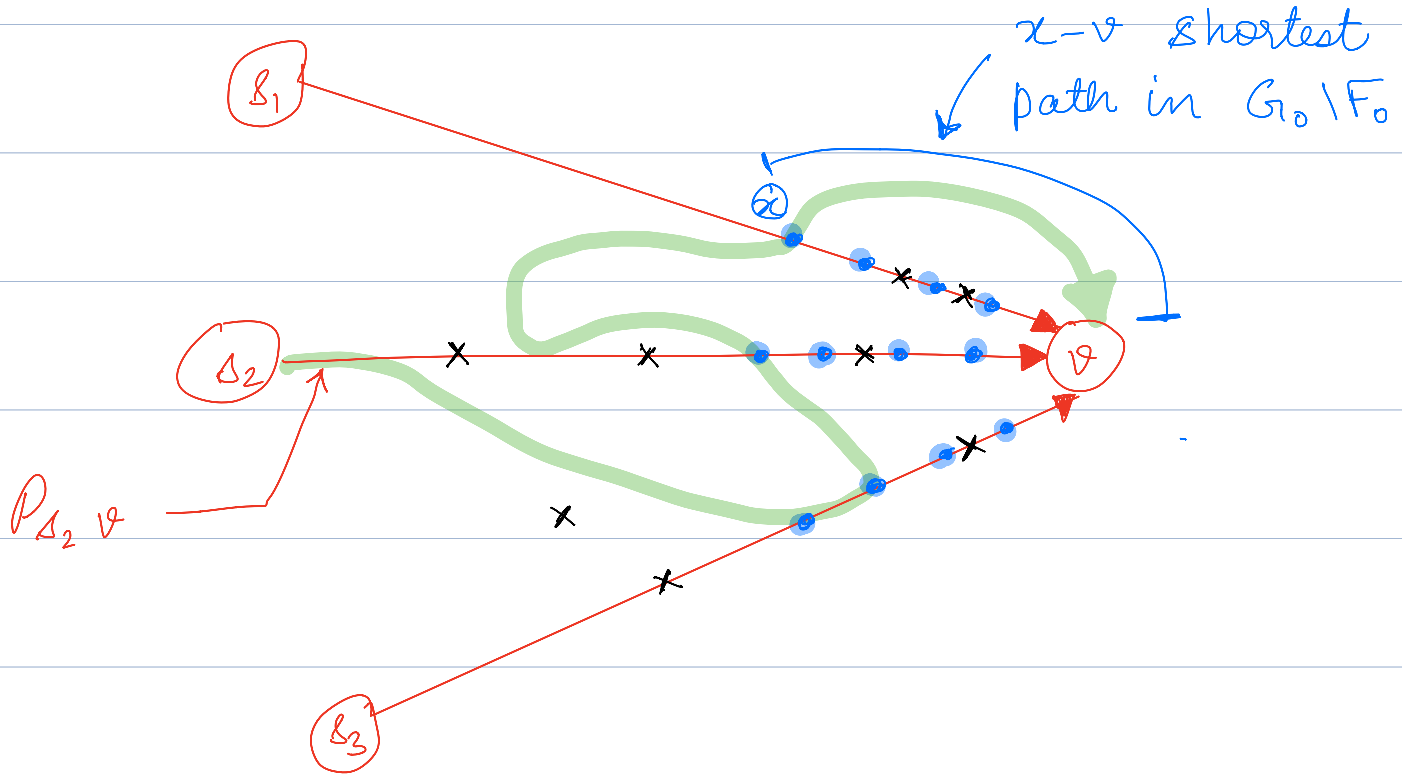
Remarks:

* $\Pi_{G \setminus F}(s,v)[x,v]$ lie in G_0

* $\Pi_{G \setminus F}(s,v)[x,v] = \Pi_{G_0 \setminus F_0}(x,v)$

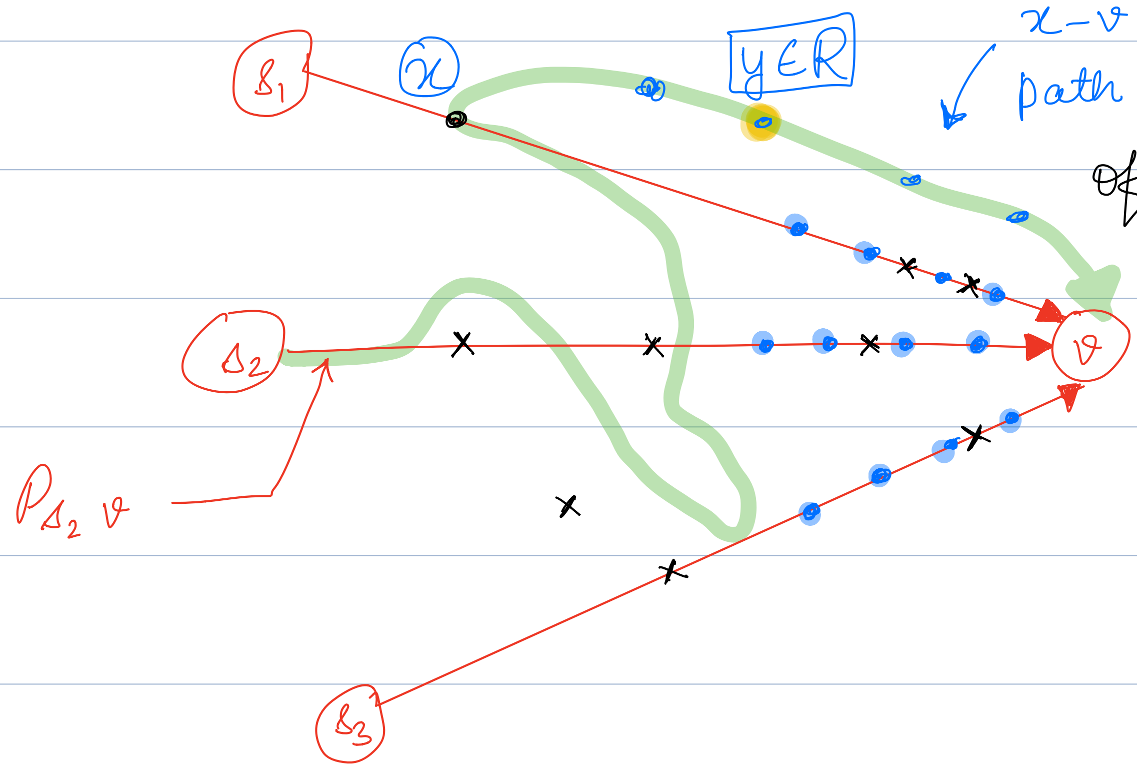
(2a) $x \in \underbrace{L}$.

$\bigcup_{s \in S}$ last-t-vertices-$P_{s,v}$



we are done bcoz $\mathcal{E}^{r}_{v}(S,G)$ contains $\mathcal{E}^{r-1}_{v}(S_0, G_0)$ and (i) $x \in S_0$ (as $x \in L$, $L \subseteq S_0$), (ii) $|F_0| \leq r-1$.

(2b) $x \notin \underbrace{h}_{\bigcup_{s \in S} \text{last-}t\text{-vertices-}P_{s,v}}$.



Remarks:

* $\Pi_{G \setminus F}(s, v)[y, v]$ lie in G_0

* $\Pi_{G \setminus F}(s, v)[y, v] = \Pi_{G_0 \setminus F_0}(y, v)$

we are done bcoz $\mathcal{E}^{r}_{v}(S, G)$ contains $\mathcal{E}^{r-1}_{v}(S_0, G_0)$

and (i) $y \in S_0$ (as $x \in R$, $R \subseteq S_0$), (ii) $|F_0| \le r-1$.

Our algorithm to compute $\mathcal{E}^r_v(S,G)$ is as below:

1. R = Random set of size $\Theta\left(\frac{n}{t} r \log n\right)$.
2. $\mathcal{L} = \{$ last t vertices of $P_{s,v} \mid s \in S \}$.
3. $\mathcal{E}^r_v(S,G) =$

$$\left(\bigcup_{s\in S} \text{LastEdge}(P_{s,v})\right) + \mathcal{E}^{r-1}_v\left(\underbrace{R\cup \mathcal{L}}_{\text{New source-set } S_0},\ \underbrace{G \setminus \bigcup_{s\in S} \text{EDGES}(P_{s,v})}_{\text{New graph } G_0}\right)$$

What is optimal value of t?

We have,

$$|S_0| = |R| + |\mathcal{L}| = \frac{n}{t} r \log n + t|S|$$

This is minimum when $\frac{n}{t} r \log n = t|S|$

or $t = \sqrt{\frac{n r \log n}{|S|}}$.

Putting this value of t gives

$$|S_0| = \sqrt{4 \, n \, r \, |S| \log n}$$

Note, $\left|\mathcal{E}_v^r(S, G)\right| = |S| + \left|\mathcal{E}_v^{r-1}(S_0, G_0)\right|$

$$\leq 2 \left|\mathcal{E}_v^{r-1}(S_0, G_0)\right|$$

Let $M(r, z)$ = Upper bound on $\mathcal{E}_v^r(S, G)$ when $|S| = z$.

This implies

$$M(r, z) \leq 2\, M\left(r-1, \sqrt{4 n r z \log n}\right)$$

Assignment 2: Solve above recurrence and obtain bound on $\mathcal{E}_v^r(S, G)$.