A deterministic construction of 1-FT-BFS

1 Definition and Notations

Definition 1. *A r-FT-BFS for a directed/undirected graph $G = (V, E)$ and a designated source $s \in V$ is a subgraph $H = (V, E_H \subseteq E)$ satisfying: for any set $F \subseteq E$ of r edges and any vertex $u \in V$,*

$$dist_{H \setminus F}(s, u) = dist_{G \setminus F}(s, u).$$

Given a graph $G = (V, E)$ on $n = |V|$ vertices and $m = |E|$ edges, and a source vertex $s \in V$, the following notations will be used.

- T: A shortest path tree of G rooted at s.
- $p(v)$: Parent of vertex v in T.
- $T[a, b]$: The unique tree-path from a to b in T.
- $T[a, b)$: The path $T[a, b] \setminus \{b\}$.
- $T(a, b)$: The path $T[a, b] \setminus \{a, b\}$.
- $P[a, b]$: The sub-path of path P lying between vertices a, b, assuming a precedes b on P.
- $P[a, b)$: The path $P[a, b] \setminus \{b\}$.
- $\text{LAST}_e(P)$: The last edge on path P.
- $P :: Q$: The path formed by concatenating paths P and Q in G. Here it is assumed that the last vertex of P is the same as the first vertex of Q.
- $x <_P y$: Representation of the information that x is a strict predecessor of y in P.

We will use the following theorem in our construction. (The theorem can be easily proved using induction).

Theorem 1. *Let $E_0 \subseteq E(G)$ be a set satisfying following: For each $w \in V(G)$ and each $F \subseteq E(G)$ of size r satisfying $dist_{G \setminus F}(s, w) \neq \infty$, there is a $s - w$ shortest path in $G \setminus F$ whose last edge lies in E_0.*

Then $G_0 = (V, E_0)$ is a r-FT-BFS of G.

2 Construction of 1-FT-BFS

For each vertex $v \in V$, we will provide construction of a set $\mathcal{E}_v$ of edges incident to v of size $O(\sqrt{n})$ such that the graph $H = \left(V, \bigcup_{v \in V} \mathcal{E}_v\right)$ is a 1-FT-BFS of G.

It suffices for us to focus on a fix vertex $v \in V$.
For each ancestor w of v in T, let $Q_{w,v}$ denote a $w - v$ shortest path in $G \setminus \big(T[s, w) \cup T(w, v)\big)$, if it exists.
We can "ensure" that the paths $Q_{w,v}$'s satisfy the following Convergence property:

Convergence Property. *For any two ancestors w_1 and w_2 of v in T, if $Q_{w_1,v}$ and $Q_{w_2,v}$ intersect at a vertex z, then the segments $Q_{w_1,v}[z,v]$ and $Q_{w_2,v}[z,v]$ are identical.*

Our set $\mathcal{E}_v$ is defined as follows:

$$\mathcal{E}_v = \big\{(p(v),v)\big\} \bigcup \big\{\textsc{last}_e(Q_{w,v}) \;\big|\; w \in T[s,v)\big\}$$

Lemma 1. $|\mathcal{E}_v| = O(\sqrt{n})$.

Proof. Let $w_1,\ldots,w_t$ be distinct ancestors of v satisfying $\textsc{last}_e(Q_{w_i,v}) \neq \textsc{last}_e(Q_{w_j,v})$, for $i \neq j$. Without loss of generality assume $w_t <_{T[s,v]} w_{t-1} <_{T[s,v]} \cdots <_{T[s,v]} w_1$. By Convergence property, for distinct indices i and j, the only common vertex between paths $Q_{w_i,v}$ and $Q_{w_j,v}$ is v. We have:

$$\begin{aligned} n-1 &\geq \sum_{i=1}^{t} \text{No. of vertices in } Q_{w_i,v} - 1 \\ &\geq \sum_{i=1}^{t} d_G(w_i,v) \\ &\geq \sum_{i=1}^{t} i \\ &= \Omega(t^2). \end{aligned}$$

Thus, $t = O(\sqrt{n})$. $\square$

Lemma 2. *For each $v \in V(G)$ and $e=(x,y) \in E(G)$, there is a $s-v$ shortest path in $G \setminus e$ whose last edge lies in $\mathcal{E}_v$.*

Proof. If $e \notin T[s,v]$, then $T[s,v]$ is a $s-v$ shortest-path in $G \setminus e$, and last edge of it trivially lies in $\mathcal{E}_v$. This is because all edges of T lie in H. So let us focus on non-trivial scenario that $e \in T[s,v]$. Let $P_{e,v}$ be $s-v$ shortest path in $G \setminus e$. Let w be last node in $P_{e,v}$ that is an ancestor of v in T.

Case 1: $w \in T[y,v)$ (i.e. w is below the failure $e=(x,y)$).

In such a case $P_{e,v}[s,w] :: T[w,v]$ is also a $s-v$ shortest path in $G \setminus e$, and the last edge on this path is $(p(v),v)$ which trivially lies in $\mathcal{E}_v$.

Case 2: $w \in T[s,x]$ (i.e. w is above the failure $e=(x,y)$).

In such a case $P_{e,v}[s,w] :: Q_{w,v}$ is also a $s-v$ shortest path in $G \setminus e$, since

1. $Q_{w,v}$ is a $w-v$ shortest path in $G \setminus \big(T[s,w) \cup T(w,v)\big)$; and
2. $P_{e,v}[w,v]$ is some $w-v$ path in $G \setminus \big(T[s,w) \cup T(w,v)\big)$

The last edge of this path also lies in $\mathcal{E}_v$. $\square$

The number of edges in H is $\sum_{v \in V} |\mathcal{E}_v|$ which by Lemma 1 is $O(n\sqrt{n})$. The correctness of H follows by Lemma 2 and Theorem 1.