Integer domination of Cartesian product graphs

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Given a graph G, a dominating set D is a set of vertices such that any vertex not in D has at least one neighbor in D. A \([k]\)-dominating multiset \(D_k\) is a multiset of vertices such that any vertex in \(G\) has at least \(k\) vertices from its closed neighborhood in \(D_k\) when counted with multiplicity. In this paper, we utilize the approach developed by Clark and Suen (2000) to prove a “Vizing-like” inequality on minimum \([k]\)-dominating multisets of graphs \(G, H\) and the Cartesian product graph \(G \square H\). Specifically, denoting the size of a minimum \([k]\)-dominating multiset as \(\gamma_{[k]}(G)\), we demonstrate that \(\gamma_{[k]}(G)\gamma_{[k]}(H) \leq 2k \gamma_{[k]}(G \square H)\).

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1. Introduction

Let \(G\) be a simple undirected graph \(G = (V, E)\) with vertex set \(V\) and edge set \(E\). The open neighborhood of a vertex \(v \in V(G)\) is denoted by \(N_G(v)\), and the closed neighborhood of \(v\) is denoted by \(N_C(v)\). A dominating set \(D\) of a graph \(G\) is a subset of \(V(G)\) such that for all \(v \in V(G)\), \(N_C(v) \cap D \neq \emptyset\), and the size of a minimum dominating set is denoted by \(\gamma(G)\). The Cartesian product of two graphs \(G\) and \(H\), denoted \(G \square H\), is the graph with vertex set \(V(G) \times V(H)\), where vertices \(gh, g'h' \in V(G \square H)\) are adjacent whenever \(g = g'\) and \((h, h') \in E(H)\), or \(h = h'\) and \((g, g') \in E(G)\).

In 1963, and again more formally in 1968, Vizing proposed a simple and elegant conjecture that has subsequently become one of the most famous open questions in domination theory.

Conjecture (Vizing [11], 1968). Given graphs \(G\) and \(H\), \(\gamma(G)\gamma(H) \leq \gamma(G \square H)\).

Over the past forty years (see [1] and references therein), Vizing’s conjecture has been shown to hold on certain restricted classes of graphs, and furthermore, upper and lower bounds on the inequality have been gradually tightened. Additionally, as numerous direct attempts on the conjecture have failed, research approaches have expanded to include explorations of similar inequalities for total, paired, and fractional domination [6]. However, the most significant breakthrough occurred in 2000, when Clark and Suen [4] demonstrated that \(\gamma(G)\gamma(H) \leq 2\gamma(G \square H)\). This “Vizing-like” inequality immediately suggested similar inequalities for total [8] and paired [9] domination (2008 and 2010, respectively). In 2011, we [3] improved the inequalities from [8,9] for total and paired domination by applying techniques similar to those of Clark and Suen, and also specific properties of binary matrices. In this paper, we explore integer domination (or \([k]\)-domination), and again generate an improved inequality with this combined technique.

A multiset is a set in which elements are allowed to appear more than once, e.g. \([1, 2, 2]\). All graphs and multisets in this paper are finite. A \([k]\)-dominating multiset \(D_k\) of a graph \(G\) is a multiset of vertices of \(V(G)\) such that, for each \(v \in V(G)\),
the number of vertices of \( N_G[u] \) contained in \( D_k \) (counted with multiplicity) is at least \( k \). A \( \gamma(k) \)-set of \( G \) is a minimum \([k]\)-dominating multiset, and the size of a minimum \([k]\)-dominating multiset is denoted by \( \gamma(k)(G) \). Additionally, note that a \([1]\)-dominating multiset is equivalent to the standard dominating set.

The notion of a \([k]\)-dominating multiset is equivalent to the more familiar notion of a \([k]\)-dominating function. The study of \([k]\)-dominating functions was first introduced by Domke, Hedetniemi, Laskar, and Fricke [5] (see also [7], pg. 90), and further explored by Brešar, Henning and Klavžar in [2]. The authors of [10] investigate integer domination in terms of graphs with specific packing numbers, and the authors of [2] prove the following “Vizing”-like inequality:

**Theorem 1** ([2]). Given graphs \( G \) and \( H \), \( \gamma(k)(G)\gamma(k)(H) \leq k(k + 1)\gamma(k)(G\square H) \).

Observe that for \( k = 1 \), Theorem 1 is equivalent to the bound proven by Clark and Suen. In this paper, we improve this upper bound from \( O(k^2) \) to \( O(k) \), and prove the following theorem:

**Theorem 2.** Given graphs \( G \) and \( H \), \( \gamma(k)(G)\gamma(k)(H) \leq 2k\gamma(k)(G\square H) \).

Again, for \( k = 1 \), Theorem 2 is equivalent to the bound proven by Clark and Suen.

In Section 2, we explain the basic notation and concepts required for the proof, and in Section 3 we present the actual proof.

## 2. Preliminaries

In this section, we introduce the necessary concepts and definitions used throughout the paper.

Given a universal set \( U \), a set \( A \) is said to be a multiset of \( U \) if its elements are only those present in \( U \). We denote the number of occurrences of a particular element \( x \) in \( A \) by \( |A|_x \). The union of multisets is denoted by \( \uplus \). Let \( A \) and \( B \) be multisets of \( U \), then \( A \uplus B \) is a multiset of \( U \) such that for each \( x \) in \( U \), \(|A \uplus B|_x = |A|_x + |B|_x \). Similarly, \( A \cap B \) is a multiset of \( U \) satisfying \( |A \cap B|_x = \min(|A|_x, |B|_x) \). The union of a multiset \( A \) with itself \( t \) times is denoted by \( \uplus^t A \). A multiset \( B \) is a submultiset of \( A \) if for each \( x \), \(|B|_x \leq |A|_x \). The cardinality of a multiset \( A \) is the summation over the number of occurrences of each element in it, i.e., \( |A| = \sum_{x \in U} |A|_x \). Finally, for multiset \( A \) and a set \( S \) subset of \( U \), \( |A|_S \) is defined as \( \sum_{x \in S} |A|_x \).

Now consider a graph \( G \). Let \( P^G = \{p_1, p_2, \ldots, p_t\} \) be a multiset whose elements are subsets of \( V(G) \). Then \( P^G \) is called a \( k \)-partition of \( V(G) \) if each vertex \( x \) in \( G \) is present in exactly \( k \) of the sets \( p_1, \ldots, p_t \). We will now see that any \( \gamma(k) \)-set of \( G \) (and a specific assignment of vertices to dominators), naturally induces a \( k \)-partition on \( V(G) \). Let \( \{u_1, \ldots, u_{\gamma(k)}\} \) be a minimum \([k]\)-dominating multiset of \( G \). To each dominator \( u_i \), we associate a subset \( P^G_i \) of \( V(G) \) as follows. Recall that for each vertex \( x \) in \( G \) there exists at least \( k \) dominators (say \( u_{i_1}, \ldots, u_{i_k} \) in \( N_G[x] \)). Therefore, by only including \( x \) in the sets \( P^G_{i_1}, \ldots, P^G_{i_k} \), the multiset \( \{P^G_{i_1}, \ldots, P^G_{i_k}\} \) is a \( k \)-partition of \( V(G) \). Additionally, note that for each \( i \), the set \( P^G_i \) is a subset of \( N_G[u_i] \).

Given a graph \( G \), we will now define the concept of domination among multisets of \( V(G) \). Given a vertex \( x \), a vertex \( y \) is said to be a dominator of \( x \) if \( y \in N_G[x] \). Let \( A, B \) be multisets of \( V(G) \). We say that \( A \) dominates \( B \) if for each \( x \in B \), \(|A|_{N_G[x]} \geq |B|_x \). In other words, the number of dominators of \( x \) in \( A \) (when counted with multiplicity) is at least the number of occurrences of \( x \) in \( B \). It is important to note that a multiset \( D \) is a \([k]\)-dominating set for \( G \) if and only if \( D \) dominates \( \uplus^k V(G) \).

**Example 1.** Consider the following graph \( G \), and let \( A = \{1, 2, 3, 4, 5, 5\} \) be a minimum 2-dominating multiset of \( G \). Assuming that vertex 1 is assigned to be dominated by vertices 1 and 2 (denoted as \( 1 \rightarrow \{1, 2\} \)), vertex 2 \( \rightarrow \{2, 3\} \), vertex 3 \( \rightarrow \{2, 3\} \), vertex 4 \( \rightarrow \{3, 4\} \) and vertex 5 \( \rightarrow \{5, 5\} \).

\[
\Phi_C(A) = \left\{ g \in V(G) \mid \text{ with } |\Phi_C(A)|_g = \sum_{h \in V(G)} |A|_{gh} \right\}
\]

Similarly \( \Phi_H(A) \) can be defined.

We end this section by stating a proposition whose proof is a straightforward application of the definitions stated above.
Proposition 1. Given graphs $G$, $H$

1. Let $A_1, B_1, A_2, B_2$ be multisets of $V(G)$. If $A_1$ dominates $B_1$, and $A_2$ dominates $B_2$, then $A_1 \cup A_2$ dominates $B_1 \cup B_2$.

2. Let $A$ be a multiset of $V(G \sqcup H)$. Then $|\Phi_G(A)| = |\Phi_H(A)| = |A|$.

3. Main proof

We now start with the details of our proof. Let $\{u_1, \ldots, u_{\gamma_k(G)}\}$ and $\{v_1, \ldots, v_{\gamma_k(H)}\}$ be minimum $\{k\}$-dominating multisets of $G$, $H$, respectively, and let $I = \{1, \ldots, \gamma_k(G)\}$ and $J = \{1, \ldots, \gamma_k(H)\}$ be the corresponding sets of indices. Additionally, let $P_G = \{P_{i_1}^G, \ldots, P_{i_{\gamma_k(G)}}^G\}$ and $P_H = \{P_{i_1}^H, \ldots, P_{i_{\gamma_k(H)}}^H\}$ be the induced $k$-partitions of $V(G)$ and $V(H)$, respectively. Finally, let $B$ be a minimum $\{k\}$-dominating multiset for graph $G \sqcup H$.

Proposition 2. Let $T \subseteq I$, $A = \cup_{i \in T} P_i^G$, and $C = \cup_{i \in T} P_i^H$. Then $A$ dominates $C$. Furthermore, for any other multiset $B$ of $(V(G), \Gamma)$, if $B$ dominates $C$, then $|B| \geq |T|$.

Proof. We first prove $A$ dominates $C$. Since $P_G^C \subseteq N_G[u_i]$, $u_i$ dominates $P_G^C$. Therefore, $\cup_{i \in T} P_i^G$ dominates $\cup_{i \in T} P_i^C$, i.e. $A$ dominates $C$. Now let multiset $W = \cup_{i \in T} \{u_i\}$. Since $B$ is any multiset dominating $C$, by Proposition 1.1 we have $B \cup W$ dominates $(\cup_{i \in T} P_i^G) \cup (\cup_{i \in T} P_i^H)$. Now as $B \cup W$ dominates $\cup_{i \in T} P_i^G$, we have a $\{k\}$-dominating multiset for graph $G$. Finally, as $A \cup W = \{u_1, \ldots, u_{\gamma_k(H)}\}$ is a $\{k\}$-set of $G$, $|B \cup W| \geq |A \cup W|$. Therefore, $|B| \geq |A| = |T|$.

Proposition 3. There exists an $\{k\}$-dominating multiset such that, for any vertex $gh$, the dominators assigned to it in each strip $(G\text{-strip} \cup H\text{-strip})$ are a subset of $D$. In other words, for each $i \in I$, $\cup_{j \in J} F(gh, P_i^G \times P_j^H)$ is a subset of $D$, and for each $j \in J$, $\cup_{i \in I} F(gh, P_i^G \times P_j^H)$ is a subset of $D$.

Proof. Consider a vertex $gh \in V(G \sqcup H)$. Let $d_0, \ldots, d_{k-1}$ be the $k$ (not necessarily distinct) dominators of $gh$ in $D$. Let $i_1, \ldots, i_k$ and $j_1, \ldots, j_k$ be indices in $I, J$, respectively, such that for $1 \leq r, s \leq k$, the block $P_{i_r}^G \times P_{j_s}^H$ contains vertex $gh$. Define $F(gh, P_{i_r}^G \times P_{j_s}^H)$ as $d_{(r+s)} \mod k$. Recall $F(gh, P_{i_r}^G \times P_{j_s}^H)$ is defined as $\emptyset$ if $gh \notin P_{i_r}^G \times P_{j_s}^H$. Thus for any index $i_r, \cup_{j \in J} F(gh, P_{i_r}^G \times P_{j_s}^H) = \{d_{(r+s)} \mod k : 0 \leq r \leq k\}$. Hence for any index $i \in I$ we have $\cup_{j \in J} F(gh, P_i^G \times P_j^H)$ is equal to $\{d_{(r+s)} \mod k : 0 \leq r \leq k\}$, and empty otherwise. This proves the first part. The proof for the second part similarly follows.

Given $gh \in P_i^G \times P_j^H$, we define $F(gh, P_i^G \times P_j^H)$ in the same way as in proof of Proposition 3. We now define a notion of $H$-dominated and $G$-dominated blocks in $P_G \times P_H$ of $V(G \sqcup H)$. A block $P_i^G \times P_j^H$ is said to be $H$-dominated if for each $h \in P_i^H$, there exists a $g \in P_i^G$ such that $F(gh, P_i^G \times P_j^H)$ belongs in the $H$-neighborhood of $gh$. Recall the $H$-neighborhood of $gh$ consists of neighbors of $gh$ (including itself) in the fiber $g$ of $V(H)$. A $G$-dominated block is defined similarly.

Let $N_i$ be the number of blocks in strip $P_i^G \times V(H)$ which are $H$-dominated, and $N_j$ be the number of blocks in strip $V(G) \times P_j^H$ which are $G$-dominated.

Proposition 4. $\sum_{i \in I} N_i + \sum_{j \in J} N_j \geq \gamma_k(G) \gamma_k(H)$.

Proof. We first show that each block is $G$-dominated, $H$-dominated or both. Consider a block $P_i^G \times P_j^H$ which is not $G$-dominated. Then there exists a vertex $g_0$ in $P_i^G$ such that for each $h \in P_j^H$, $F(g_0h, P_i^G \times P_j^H)$ does not lie in the $G$-neighborhood of $g_0h$. Suppose $P_i^G \times P_j^H$ is also not $H$-dominated. Then there will exist a vertex $h_0$ in $P_j^H$ such that for each $g \in P_i^G$, $F(gh_0, P_i^G \times P_j^H)$ does not lie in the $H$-neighborhood of $gh_0$. But this means that $F(gh_0h, P_i^G \times P_j^H)$ lies neither in the $G$-neighborhood nor $H$-neighborhood of $g_0h$. This is a contradiction.

Now we give an upper bound on the total number of $G$-dominated and $H$-dominated blocks.

Proposition 5. $\sum_{i \in I} N_i + \sum_{j \in J} N_j \leq k \gamma_k(G \sqcup H)$. 

\[\sum_{i \in I} N_i + \sum_{j \in J} N_j \leq k \gamma_k(G \sqcup H)\]
Proof. We prove that $\sum_{i \in I} N_i \leq k_{Y}(G \square H)$. The proof for $\sum_{j \in J} N_j \leq k_{Y}(G \square H)$ follows similarly. For $i \in I$, define

$$Y_i = \{p^H \mid p^C \times p^H \text{ is } H\text{-dominated}\}$$

$$S_i = D \cap \{p^H \times V(H)\}.$$  

We now divide the proof in three parts.

**Claim 1.** For each $i \in I$, $\Phi_i(S_i)$ dominates $Y_i$.

**Proof.** For any fixed $i \in I$ consider a vertex $h \in Y_i$. Let it have $\alpha$ occurrences in $Y_i$. This means there are $\alpha$ blocks $p^C \times p^H_{j_1}, \ldots, p^C \times p^H_{j_\alpha}$ which are $H$-dominated and $h$ belongs in each of $p^H_{j_t}$, $1 \leq t \leq \alpha$. Therefore, for each $t$, there exists a vertex $g_t$, $h_t \in p^C \times p^H_{j_t}$ such that $F(g_t, h_t, p^C \times p^H_{j_t})$ belongs in $H$-neighborhood of $g_t, h_t$. Let this dominator $F(g_t, h_t, p^C \times p^H_{j_t})$ be $g_t, h_t$. Then, $h_1, \ldots, h_\alpha$ must lie in $N_h[h]$. We will show that $\{g_1, h_1, \ldots, g_\alpha, h_\alpha\}$ is a subset of $D$.

Let $\beta$ be the number of distinct elements in $\{g_1, h_1, \ldots, g_\alpha, h_\alpha\}$ (i.e. when counted without multiplicity). Let $L_1, \ldots, L_\beta$ be a partition of $[\beta]$ such that for $t, t'$ lying in same $L_i$, $g_t = g_{t'}$ and for $t, t'$ lying in different $L_i$'s $g_t \neq g_{t'}$. Now from **Proposition 3** we have that $\cup_{i \leq j} F(g_t, h, p^C \times p^H_{j_t})$ is a subset of $D$. Thus, for each $r, \cup_{i \leq j} g_t, h_t$ is a subset of $D$. Also for $r \neq s$, the intersection of $\cup_{i \leq j} g_t, h_t$ and $\cup_{i \leq j} g_t, h_t$ is empty. This is because if $g_{t_r} \neq g_{t_s}$ then $g_{t_r}, h_{t_r}$ will belong in different $G$-fibers. Hence, $\cup_{i \leq j} g_t, h_t$ is a subset of $D$.

Now note that since $\{g_1, h_1, \ldots, g_\alpha, h_\alpha\}$ is a subset of $D$ and each $g_t, h_t$ lies in strip $p^C \times V(H)$, we have that $\{h_1, \ldots, h_\alpha\}$ is a subset of $\Phi_i(S_i)$. Hence, there exist at least $\alpha$ dominators for vertex $h$ in $\Phi_i(S_i)$ (when counted with multiplicity). This proves $\Phi_i(S_i)$ dominates $Y_i$. □

**Claim 2.** For each $i \in I$, $N_i \leq |\Phi_i(S_i)|$.

**Proof.** Let $T_i = \{j \mid p^C \times p^H_j \text{ is } H\text{-dominated}\}$. Then $Y_i = \cup_{j \in T_i} p^H_j$. Now since $\Phi_i(S_i)$ dominates $Y_i$ from **Proposition 2** we have that $|\Phi_i(S_i)| \geq |T_i| = N_i$. □

**Claim 3.** The multiset $\cup_{i \leq j} S_i$ is equal to $\cup^B D$.

**Proof.** Consider any $gh \in D$. Let $n_0$ be the number of occurrences of $gh$ in $D$. Now consider any $p^C \in P^C$. If $p^C$ contains $g$, then $\cup^B (p^C \times V(H))$ will contain $k_0$ copies of $gh$, and $S_i$ will contain $n_0$ copies of $gh$. If $g \notin P^C$, then $S_i$ will not contain zero copies of $gh$. Hence there are exactly $k S_i$'s which have $n_0$ copies of $gh$, and all the remaining do not contain vertex $gh$. Additionally, since $\cup_{i \leq j} S_i$ contains $k_0$ copies of $gh$ we see $\cup_{i \leq j} S_i = \cup^B D$. □

Finally, $\sum_{i \in I} N_i \leq \sum_{i \in I} |\Phi_i(S_i)| = |\cup_{i \leq j} \Phi_i(S_i)| = |\cup_p D| = k_{Y}(H)$. Thus the result follows. □

Finally from **Propositions 4** and 5 we get $k_{Y}(G \square H) \leq 2k_{Y}(G \square H)$.

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