Extremal Distance Spanners

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(joint work with Omer Gold)
Graph Sparsification

Deals with sparsifying a graph while preserving certain properties “approximately”

eg. Distances
eg. Pair-wise Cuts

G

Sparse subgraph
Distance Spanners

Definition [PU’89, PS’89]:

Given a graph $G = (V, E)$, a spanner is a sub-graph $H = (V, E_H)$ which is

1. sparse
2. preserves pair-wise distances approximately.

$$\text{dist}_H(x, y) \leq t \text{ dist}_G(x, y)$$

$t$: stretch of the spanner

Example:

![Example diagram showing a graph and a spanner with stretch = 3]
Distance Spanners

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$t$: stretch of the spanner

Tradeoff (undirected):

- stretch = 3, size = $O(n^{1+1/2})$
- stretch = 5, size = $O(n^{1+1/3})$
- stretch = 7, size = $O(n^{1+1/4})$
- stretch = $2k-1$, size = $O(n^{1+1/k})$

$n=|V|=\text{number of vertices}$
Sparseness for Directed graphs?

Sparseness is NOT always possible

Directed Complete bi-partite

2n vertices
n^2 edges

Removing single edge \((b_i, c_j)\) increases STRETCH to infinity!
Sparseness for Directed strongly-connected graphs?

Removing edge \((b_i, c_j)\) increases STRETCH to \(O(\text{diameter})\).
Extremal Distance Spanners

Sparisfy a graph to *preserve* EXTREMAL/LARGE distances
Extremal Distances

**Diameter**

\[ Diam(G) = \text{maximum} \ dist_G(x,y) \]

\[ x,y \]

**Eccentricity**

\[ Ecc(x;G) = \text{maximum} \ dist_G(x,y) \]

\[ y \]

diameter = 3
eccentricity(c) = 2
eccentricity(f) = 3
Diameter Spanners

A sub-graph $H=(V,E_H)$ of $G=(V,E)$ which is

1. sparse
2. preserves diameter approximately.

$$Diam(H) \leq \lceil t \ Diam(G) \rceil$$

$t$: stretch of the diameter-spanner

Eccentricity Spanners

A sub-graph $H=(V,E_H)$ of $G=(V,E)$ which is

1. sparse
2. preserves eccentricities approximately.

$$Ecc(x;H) \leq \lceil t \ Ecc(x;G) \rceil$$

$t$: stretch of the eccentricity-spanner
Diameter Spanners
Diameter Spanners: Trivial construction

\[ H = \text{OUT-BFS}(a) \cup \text{IN-BFS}(a) \text{ is a diameter spanner with} \]

- Stretch = 2
- Size = \( O(n) \)

Can we get stretch better than 2?
### Undirected Graphs: Various kind of spanners known

<table>
<thead>
<tr>
<th>Type</th>
<th>Definition</th>
<th>Some Known Results</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Multiplicative:</td>
<td>$dist_H(x,y) \leq (2k-1) dist_G(x,y)$</td>
<td>$Size = O(n^{1+1/k})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2. Additive:</td>
<td>$dist_H(x,y) \leq dist_G(x,y) + \beta$</td>
<td>$Size = O(n^{4/3})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\beta = 6$</td>
</tr>
<tr>
<td>3. $(1+\varepsilon, \beta)$:</td>
<td>$dist_H(x,y) \leq (1+\varepsilon) dist_G(x,y) + \beta$</td>
<td>$Size = O(n^{1+1/k})$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\beta = func(k, \varepsilon)$</td>
</tr>
<tr>
<td>4. Sublinear:</td>
<td>If $dist_G(x,y) = d$, then $dist_H(x,y) \leq d + o(d)$</td>
<td>$Size = O(n^{1+1/k})$</td>
</tr>
<tr>
<td></td>
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</tr>
</tbody>
</table>

**Good t-diameter spanner for graphs with large diameter**

$O(d^{1-1/(k-1)})$
NEW Results on Diameter-spanner (directed graph)

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Size (Edges)</th>
<th>Computation time</th>
<th>Lower Bounds</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5</td>
<td>$O(n^{1.5})$</td>
<td>$O(m\sqrt{n})$</td>
<td>$D &lt; n^{1/4}$</td>
</tr>
<tr>
<td>$1.5 + o(1)$</td>
<td>$O(n\sqrt{n/D})$</td>
<td>$O(m\sqrt{n/D})$</td>
<td>$D &lt; n^{1/3}$</td>
</tr>
<tr>
<td>$5/3$</td>
<td>$O(n^{4/3}D^{1/3})$</td>
<td>$O(m(nD)^{2/3})$</td>
<td>$D &lt; n^{0.1}$</td>
</tr>
<tr>
<td>$5/3 + o(1)$</td>
<td>$O(n^{4/3}D^{-1/3})$</td>
<td>$O(m(n/D)^{2/3})$</td>
<td>$D &lt; n^{1/7}$</td>
</tr>
</tbody>
</table>

Techniques (for 5/3 stretch): Inspired by works on diameter-approximation
A simple \((3/2)\)-diameter-spanner with \(O(n^{1.5})\) edges

Algorithm:

Initialize \(H = (V, \emptyset)\).

1. For each \(v \in V\), add to \(H\)
   - arbitrary \(\sqrt{n}\) incoming edges of \(v\), and
   - arbitrary \(\sqrt{n}\) outgoing edges of \(v\).

2. Take a random set \(R\) of \(\tilde{O}(\sqrt{n})\) vertices.

3. For each \(s \in R\), add \(IN-BFS(s)\) and \(OUT-BFS(s)\) to \(H\).

Correctness:

\[
\begin{align*}
\text{case 1:} & \quad \text{Low out-degree} \quad \text{Low in-degree} \\
& \quad \text{Entire path lies in } H
\end{align*}
\]
A simple $(3/2)$-diameter-spanner with $O(n^{1.5})$ edges

Algorithm:

Initialize $H = (V, \emptyset)$.

1. For each $v \in V$, add to $H$
   - arbitrary $\sqrt{n}$ incoming edges of $v$, and
   - arbitrary $\sqrt{n}$ outgoing edges of $v$.

2. Take a random set $R$ of $\tilde{O}(\sqrt{n})$ vertices.

3. For each $s \in R$, add $IN$-$BFS(s)$ and $OUT$-$BFS(s)$ to $H$.

Correctness:

\[ dist_H(x, y) \leq \lceil 1.5 \text{Diam}(G) \rceil \]
Lower Bound: For every $n$ and every $D$, there exists a directed unweighted graph $G$ with $n$ vertices and diameter $D$, such that any diameter-spanner of $G$ with "< 3/2" stretch requires $\Omega(n^2/D^2)$ edges.
Trivial constructions for Eccentricity Spanners

If ‘a’ is the centre,

\[ H = \text{OUT-BFS}(a) + \text{IN-BFS}(a) \] is eccentricity spanner with

- Stretch = 2
- Size = \( O(n) \)

**Centre**: \( \text{arg min}_x \text{ecc}(x; G) \)

**radius(G)**: \( \text{min}_x \text{ecc}(x; G) \)
Eccentricity-spanner

Can we achieve in $O(m)$ time an $O(n)$ size “eccentricity-spanner” of “at most 2” stretch?

No $o(\min\{mn, n^\omega\})$ time algorithm known for computing Graph Centre!
Pseudo-centre (key ingredient to Eccentricity-spanner)

**Pseudo-Centre:** A set $U$ of vertices such that $\text{dist}(U,w) \leq \text{radius}(G)$, for each $w$.

**Theorem:** In $O(m \log^2 n)$ time for any given directed weighted graph $G=(V,E)$ we can compute “Pseudo-Centre” for $G$ of $O(\log^2 n)$ size.
Initialize $X=V$, and pseudo-centre $U$ to empty-set.

While $|X| > 0$:

1. Take a random subset $R$ of $X$ of $O(\log n)$ vertices, and add it to $U$.

2. Discard (roughly 50%) vertices from $X$ that we can verify efficiently to have eccentricity larger than out-ecc($R$).

Algorithm sketch:
Eccentricity Spanner: Almost linear time construction

Upper Bound: We can compute in $O(m \log^2 n)$ expected time for any directed weighted graph an eccentricity-spanner of stretch 2 that contains just $O(n \log^2 n)$ edges.

Lower Bound: For every $n$ and every $\rho$, there exists an unweighted directed graph $G$ with $n$ vertices and radius $\rho$, such that any eccentricity-spanner of $G$ with “$<2$” stretch requires $\Omega(n^2/\rho^2)$ edges.

Question: Better than 2.. ??
By-product of Pseudo-centre:
Eccentricity and Radius approximation

<table>
<thead>
<tr>
<th>Stretch</th>
<th>Time</th>
<th>Parameter</th>
<th>Graph</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$\tilde{O}(m\sqrt{n})$</td>
<td>Radius</td>
<td>unweighted</td>
<td>[AVW16]</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{O}(m\sqrt{n})$</td>
<td>Eccentricities</td>
<td>weighted</td>
<td>[BRSVW18]</td>
</tr>
<tr>
<td>$2 + \delta$</td>
<td>$\tilde{O}(m/\delta)$</td>
<td>Eccentricities</td>
<td>weighted</td>
<td>[BRSVW18]</td>
</tr>
<tr>
<td>2</td>
<td>$\tilde{O}(m)$</td>
<td>Eccentricities</td>
<td>weighted</td>
<td>New</td>
</tr>
</tbody>
</table>
Future Research
Future Research

• Complete size-stretch trade-off for diameter spanners.

• Eccentricity spanners complete tradeoff for graphs with small radius.