On Efficiently Realizing Interval Sequences

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“Graph Realization” ???

Given certain properties for a graph, say - degrees, flow, cuts, eccentricities, connectivity, etc.

aim is to characterise if the realizing graph exists.
“Graph Realization” ???

Why bother to learn this domain?

- Study of graphs and networks is indispensable.

- To better understand graphs and networks, it is essential to understand if a graph of certain property even exists.
# Graphic Sequence

## Degree Sequence

- Non-increasing sequence of vertex degrees of a graph.
- \( D = \text{deg}(G) = (3, 3, 3, 3, 3, 5) \)

## Graphic Sequence

- Sequence of integers which form degree sequence of some graph.
- Examples:
  - No - \( D = (3, 5) \)
  - Yes - \( D = (3, 3, 3, 3) \)

Also known as "realizable sequence"!
### Some Well-known Work

<table>
<thead>
<tr>
<th>Work</th>
<th>Time Complexity</th>
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<td>Erdös and Gallai:</td>
<td>O(n) time</td>
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<td>Characterisation of graphic sequences.</td>
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<td>Havel and Hakimi:</td>
<td>O(sum(D)) time</td>
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<td>Finding a realizing graph for graphic sequences.</td>
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### Realizable Interval Sequence

<table>
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<th>Interval Sequence</th>
<th>Realizable Interval Sequence</th>
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| • Sequence of $n$ intervals  
  $S = ([a_1, b_1], \ldots, [a_n, b_n])$,  
  
  • Alternate Notation:  
  $S=(A, B)$  
  where, $A=(a_1,\ldots,a_n)$ & $B=(b_1,\ldots,b_n)$  | • $S=(A, B)$ is realizable if there is graphic $n$-length sequence $D$ such that $A \leq D \leq B$.  
  
  • Examples:  
  
    - **No** - $S = ([1,2], [99,100])$  
    - **Yes** - $S = ([1,3], [1,3], [1,3])$ |
1. Verification:
   Find an efficient algorithm for verifying the realizability of any given interval-sequence $S$?

2. Certificate:
   Given a realizable interval-sequence $S$, compute a certificate (i.e. a graphic sequence) realizing it.
## Existing Work

Some work done in the recent years:

<table>
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<tr>
<th>Reference</th>
<th>Description</th>
<th>Time Complexity</th>
</tr>
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<tbody>
<tr>
<td>Cai et al. (2000)</td>
<td>Characterisation of realizable interval sequences.</td>
<td>Verification in O(n) time</td>
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<tr>
<td>Hell and Kirkpatrick (2009)</td>
<td>Finding graphic sequence, if exists, if not, then finds a sequence with least L1 deviation.</td>
<td>Certification in O(sum(B)) time</td>
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<tr>
<td>Garg et al. (2011)</td>
<td>Constructive proof of characterisation of Cai et al. (as byproduct finds graphic certificate).</td>
<td>Certification in O(sum(B)) time</td>
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Can we find certificate in linear/near-linear time?
Our Contributions

1. **Certificate computation:**
   There exists an algorithm that for any \( n \geq 1 \) and any \( n \)-length interval sequence \( S \), computes a graphic sequence \( D \) realizing \( S \), if exists, in \( O(n \log n) \) time.

2. **Minimum-deviation graphic sequence:**
   For non-realizable interval sequence \( S \), we can outputs in \( O(n \log n) \) time a graphic sequence \( D \) minimizing the deviation \( \delta(D, S) \).

3. **Regular-most certificate:**
   There exists an algorithm that for any \( n \geq 1 \) and any \( n \)-length interval sequence \( S \), computes the most regular graphic sequence realizing it, if exists, in \( O(n^2) \) time.
Deviation Measure

- Deviation of sequence:

\[
\delta(D, S) = \sum_{i=1}^{n} \max\{0, (a_i - d_i), (d_i - b_i)\}
\]

Three terms capture: (i) no-deviation, (ii) lower-deviation, (iii) upper-deviation.
Irregularity Measure

- **Spread** of sequence:

\[
\phi(D) = \sum_{i \neq j} |d_i - d_j|
\]
Some Simple Definitions/Properties
Levelling operation

Levelling operation $\pi(D, \alpha, \beta)$:
A sequence obtained from $D=(d_1,\ldots,d_n)$ by
- adding $-1$ to $d_\alpha$ and $+1$ to $d_\beta$, if $d_\alpha > d_\beta$.
- adding $-1$ to $d_\beta$ and $+1$ to $d_\alpha$, if $d_\beta > d_\alpha$.

Levelling Property:
If $D$ is graphic, then $\pi(D, \alpha, \beta)$ is also graphic.

Reasoning: A neighbour of the higher degree vertex can be transferred to vertex of lower degree!
Volume of Sequence

Volume of $D$ with respect to $S=(A,B)$: $L_1(A,D) = \text{sum}(D-A)$.

(Fact: Levelling operation preserves volume.)
Levelled Sequence

Levelled Sequence with respect to $S=(A,B)$:
A sequence $D \in (A,B)$ on which any levelling operation $\pi$ ensuring $\pi(D,\alpha,\beta)$ lies in $(A,B)$, only permutes its elements.

Existence reasoning?: Levelling operation on a sequence $D$ that results in a non-similar sequence, reduces its spread $\phi(D)$. 
Example: A Levelled Sequence $D = (12, 10, 9, 8, 8, \ldots)$
Example: A Levelled Sequence of volume “33”
Count of Levelled Sequence

There are at most $O(n^2)$ levelled sequences that are non-similar, each having different volume.
Certification Algorithm
Theorem: Given any volume $L$ satisfying $L \in [0, \text{sum}(B-A)]$, we can compute a levelled sequence $D \in (A,B)$ of volume $L$ in $O(n)$ time.

Algorithm

1. For each $L \in [0, \text{sum}(B-A)]$, compute a levelled sequence $D_L$ of volume $L$.

2. Check using Erdős and Gallai characterisation whether $D_L$ is graphic, for each $L$. 

A Simple $O(n^3)$ Certification Algorithm
Strategy: Divide and Conquer

Using further properties of levelled sequences, we obtain...

**Theorem:** For any realizable interval sequence $S = (A,B)$, and any levelled sequence $C \in S$,

- either the interval-sequence $(A,C)$, or
- the interval-sequence $(B,C)$, is realizable.
An $O(n \log n)$ Certification Algorithm

**Algorithm**

1. Initialize $(A_0, B_0)$ to $(A, B)$;

2. While $L_1(A_0, B_0) \geq 2$ do
   i. $C_0 \leftarrow$ A levelled sequence of volume $\lceil L_1(A_0, B_0)/2 \rceil$;
   ii. If (Interval-sequence $(A_0, C_0)$ is realizable) then $B_0 \leftarrow C_0$;
   iii. Else $A_0 \leftarrow C_0$;

3. If $A_0$ is graphic then Return $A_0$, else Return $B_0$;

We crucially use here Cai et al. characterisation for realizable interval sequences that takes $O(n)$ time.
An $O(n \log n)$ Certification Algorithm

**Theorem:** There exists an algorithm that for any $n \geq 1$ and any $n$-length interval sequence $S$, computes a graphic sequence $D$ realizing $S$, if exists, in $O(n \log n)$ time.
Open Questions

(i) Improving the time to $O(n)$?

(ii) Answering the question for directed graphs in linear/near-linear time?

(iii) For non-realizable interval sequences, finding graphic sequence minimizing deviations with respect to general $L_k$ norm?