Graph Realization:
Maximum Degree in Vertex Neighborhoods

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What is Graph Realization?

Given certain properties for a graph, example: Degrees, Max-Flow, Minimum-Cuts, Distances, Connectivity, etc. find if graph realizing the given property.
Classical Work: Degree Sequence Realizability

Given a sequence $D = (d_1, d_2, \ldots, d_n)$ of $n$ integers.

Find if there is a graph $G$ with degree-sequence $D$.

<table>
<thead>
<tr>
<th>Well Known Results</th>
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<tbody>
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<td><strong>Erdős and Gallai (1960):</strong></td>
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<td>(Characterisation)</td>
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<td><strong>Havel and Hakimi (1955, 1962):</strong></td>
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<td>(Finding a realization)</td>
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Maximum-Neighborhood-Degree (MaxNDeg)

Input: \( \sigma = (d^n_{\ell}, \cdots, d^n_1) \) with
- \( d_{\ell} > \cdots > d_1 \geq 1 \),
- \( n_i \geq 1 \)

\( \sigma = (4^5, 3^2, 2^1) \)

Find corresponding \( G \) (if exists)
Necessary conditions

\[ \sigma = (d_{n_\ell}^{n_\ell}, \ldots, d_1^{n_1}) \]

1. \( n_\ell \geq d_\ell + 1 \),
2. If \( d_1 = 1 \), then \( n_1 \) is even

Are these sufficient?
Sub-structure Property

\[ \sigma = (d_{n_\ell}^{n_\ell}, \ldots, d_{n_i}^{n_i}, \ldots, d_{n_1}^{n_1}) \]

is MaxNDeg-realizable

\[ \hat{\sigma} = (d_{n_\ell}^{n_\ell}, \ldots, d_{n_i}^{n_i}) \]

is MaxNDeg-realizable.

Proof Idea:

- Let \( G = (V, E) \) be a realization of \( \sigma \), and \( W_i \) be vertices of degree at least \( d_i \)
- Induced subgraph \( G[W_i] \) is realization of \( \hat{\sigma} \)
A Constructive Sufficiency Proof
(Based on substructure property)

Valid list in a graph:
$L = (a_1, \ldots, a_t)$ satisfying
• $\deg(a_i) \leq i$, and
• No edges within $L$

Oracle

A realization $G_i$ of $(d_{n_1}^{d_\ell}, \ldots, d_{n_i}^{d_\ell})$
containing valid list of size $(d_i - 2)$

Realization $G_{i-1}$ of $(d_{n_1}^{d_\ell}, \ldots, d_{n_i}^{d_\ell}, d_{i-1}^{d_\ell})$
containing valid list of size $(d_{i-1} - 2)$

BASE CASE: Compute
a realization $G_\ell$ of $d_\ell^{n_\ell}$
containing a valid list
of size $d_\ell - 2$. 
A Constructive Sufficiency Proof
(Based on substructure property)

A realization $G_i$ of $(d_{i}^{n_{i}}, \cdots, d_{i}^{n_{i}})$ containing valid list of size $(d_i - 2)$

Oracle

Realization $G_{i-1}$ of $(d_{i-1}^{n_{i}}, \cdots, d_{i-1}^{n_{i}}, d_{i-1}^{n_{i-1}})$ containing valid list of size $(d_{i-1} - 2)$

Why $n_{\ell} \geq d_{\ell} + 1$ ?

BASE CASE: Compute a realization $G_{\ell}$ of $d_{\ell}^{n_{\ell}}$ containing a valid list of size $d_{\ell} - 2$. 

$$G_i \leftarrow n_{i-1} \text{ nodes}$$
A Constructive Sufficiency Proof
(Based on substructure property)

A realization $G_i$ of $(d_i^{n_\ell}, \ldots, d_i^{n_i})$ containing valid list of size $(d_i - 2)$

Oracle

Realization $G_{i-1}$ of $(d_i^{n_\ell}, \ldots, d_i^{n_i}, d_{i-1}^{n_{i-1}})$ containing valid list of size $(d_{i-1} - 2)$

Why If $d_1 = 1$, implies $n_1$ is even?

BASE CASE: Compute a realization $G_{\ell}$ of $d_\ell^{n_\ell}$ containing a valid list of size $d_\ell - 2$. 

$G_i$

$L$
Characterisation of MaxNDeg profiles

<table>
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<th>Our Results</th>
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| **Connected graphs** | $d_\ell \leq n_\ell - 1$
  | If $d_1 = 1$, then $\sigma = (1^2)$ |
| **General graphs** | $d_\ell \leq n_\ell - 1$
  | If $d_1 = 1$, then $n_1$ is even |
What about Exclusive neighbourhood!

Can we characterize the profiles that are exclusive-MaxNDeg realizable?

\[ \sigma = (4^4, 3^3, 2^1) \]

The previous two conditions are NOT sufficient and necessary for exclusive MaxNDeg realization.
## Characterisation of Exclusive-MaxNDeg profiles

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| **Connected graphs** | $d_\ell \leq \min\{n_\ell, n - 1\}$  
|        | $d_1 \geq 2 \text{ or } \sigma = (d^d, 1^1) \text{ or } \sigma = (1^2)$  
|        | $\sigma \neq (d_\ell^{d_\ell+1}, 2^1)$  |
| **General graphs** | $\sigma_1 :$ connected Exclusive-MaxNDeg realization.  
|        | $\sigma_2 = (1^{2\alpha}) \text{ or } \sigma_2 = (d^d, 1^{2\alpha+1})$, for $d \geq 2, \alpha \geq 0$.  |
Number of Realizable MaxNDeg Profiles??

Number of non-increasing sequences of length $n$ with values $[1, n-1]$ is $\Theta(4^n/\sqrt{n})$.

Number of realizable MaxNDeg profiles for $n$-vertex graphs is $\Theta(2^n)$.
Future Work

- Generating a random graph realization of a given profile \((d_{\ell}^{n_{\ell}}, \cdots, d_1^{n_1})\)

- Complete characterization of Minimum-Neighborhood-Degree profiles