Online Dominating Set and Coloring ⋆

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Abstract. In this paper, we present online deterministic algorithms for minimum coloring, minimum dominating set and its variants in the context of geometric intersection graphs. We consider a graph parameter: the independent kissing number ζ, which is a number equal to 'the size of the largest induced star in the graph -1'. For a graph with an independent kissing number at most ζ, we obtain an algorithm having an optimal competitive ratio of ζ, for the minimum dominating set and the minimum independent dominating set problems; however, for the minimum connected dominating set problem, we obtain a competitive ratio of at most 2ζ. In addition, we prove that for the minimum connected dominating set problem, any deterministic online algorithm has a competitive ratio of at least $2(\zeta-1)$ for the geometric intersection graph of translates of a convex object in $\mathbb{R}^2$. Next, for the minimum coloring problem, we present an algorithm having a competitive ratio of $O\left(\zeta' \log m\right)$ for geometric intersection graphs of bounded scaled α-fat objects in $\mathbb{R}^d$ having a width in between $[1, m]$, where ζ' is the independent kissing number of the geometric intersection graph of bounded scaled α-fat objects having a width in between $[1, 2]$. Finally, we investigate the value of ζ for geometric intersection graphs of various families of geometric objects.

Keywords: α-Fat objects · Coloring · Connected dominating set · Dominating set · Independent kissing number · t-relaxed coloring.

1 Introduction

We consider online algorithms for some well-known NP-hard problems: the minimum dominating set problem and its variants and the minimum coloring problem. Dominating set and its variants have several applications in wireless ad-hoc networks, routing, etc. [3,4]; while coloring has diverse applications in frequency assignment, scheduling and many more [1,14,15].

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Minimum Dominating Set and its Variants. For a graph $G = (V, E)$, a subset $D \subseteq V$ is a *dominating set* (DS) if for each vertex $v \in V$, either $v \in D$ (containment) or there exists an edge $\{u, v\} \in E$ such that $u \in D$ (dominance). A dominating set D is said to be a *connected dominating set* (CDS) if the induced subgraph $G[D]$ is connected (if G is not connected, then $G[D]$ must be connected for each connected component of G). A dominating set D is said to be an *independent dominating set* (IDS) if the induced subgraph $G[D]$ is an independent set. The *minimum dominating set* (MDS) problem involves finding a dominating set of the minimum cardinality. Similarly, the objectives of the *minimum connected dominating set* (MCDS) problem and the *minimum independent dominating set* (MIDS) problem are to find a CDS and IDS, respectively, with the minimum number of vertices.

Throughout the paper, we consider online algorithms for *vertex arrival model* of graphs where a new vertex is revealed with its edges incident to previously appeared vertices. The dominating set and its variants can be considered in various online models [4]. In *Classical-Online-Model* (also known as "Standard-Model"), upon the arrival of a new vertex, an online algorithm must either accept the vertex by adding it to the solution set or reject it. In *Relaxed-Online-Model* (also known as "Late-Accept-Model"), upon the arrival of a new vertex, in addition to the revealed vertex, an online algorithm may also include any of the previously arrived vertices to the solution set. Note that once a vertex is included in the solution set for either model, the decision cannot be reversed in the future. For the MCDS problem, if we cannot add previously arrived vertices in the solution set, the solution may result in a disconnected dominating set [3]. Therefore, in this paper, for the case of the MCDS problem, we use Relaxed-Online-Model; while for MDS and MIDS, we use Classical-Online-Model. In addition, the revealed induced subgraph must always be connected for the MCDS problem.

Minimum Coloring Problem. For a graph $G = (V, E)$, the *coloring* is to assign colors (positive integers) to the vertices of G. The *minimum coloring* (MC) problem is to find a coloring with the minimum number of distinct colors such that no two adjacent vertices (vertices connected by an edge) have the same color. In the online version, upon the arrival of each new vertex v, an algorithm needs to immediately assign to v a feasible color, i.e., one distinct from the colors assigned to the neighbours of v that have already arrived. The color of v cannot be changed in future.

We analyze the quality of our online algorithm by competitive analysis [2]. An online algorithm *ALG* for a minimization problem is said to be *c-competitive*, if there exists a constant d such that for any input sequence $\mathcal{I}$, we have $\mathcal{A}(\mathcal{I}) \leq c \times \mathcal{O}(\mathcal{I}) + d$, where $\mathcal{A}(\mathcal{I})$ and $\mathcal{O}(\mathcal{I})$ are the cost of solutions produced by *ALG* and an optimal offline algorithm, respectively, for the input $\mathcal{I}$. The smallest c for which *ALG* is c-competitive is known as an *asymptotic competitive ratio* of *ALG* [3]. The smallest c for which *ALG* is c-competitive with $d = 0$ is called an *absolute competitive ratio* (also known as strict-competitive ratio) of *ALG* [3].

If not explicitly specified, we use the term "competitive ratio" to mean absolute competitive ratio.

1.1 Preliminaries

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$, where n is a positive real number. In this paper, we focus on geometric intersection graphs due to their applications in wireless sensors, network routing, medical imaging, etc. For a family $\mathcal{S}$ of geometric objects in $\mathbb{R}^d$, the *geometric intersection graph* G of $\mathcal{S}$ is an undirected graph with set of vertices same as $\mathcal{S}$, and the set of edges is defined as $E = \{\{u, v\} | u, v \in \mathcal{S}, u \cap v \neq \emptyset\}$. Several researchers have used the kissing number as a parameter to give an upper or lower bound for geometric problems. For instance, Butenko et al. [6] used it to prove the upper bound of the MCDS problem in the offline setup for unit balls in $\mathbb{R}^3$, whereas Dumitrescu et al. [12] used it to prove the upper bound for the unit covering problem for balls in $\mathbb{R}^d$. Similar to kissing number, we use a graph parameter- independent kissing number ζ. Let $\varphi(G)$ denote the size of a maximum independent set of a graph $G = (V, E)$. For any vertex $v \in V$, let $N(v) = \{u(\neq v) \in V | \{u, v\} \in E\}$ be the neighbourhood of the vertex v. Now, we define the *independent kissing number* ζ for graphs.

Definition 1 (Independent Kissing Number). *The independent kissing number ζ of a graph $G = (V, E)$ is defined as* $\max_{v \in V}\{\varphi(G[N(v)])\}$.

Note that the independent kissing number equals 'the size of the largest induced star in the graph -1'. In other words, a graph with independent kissing number ζ is a $K_{1,\zeta+1}$-free graph. Moreover, the value of ζ may be very small compared to the number of vertices in a graph. For example, the value of ζ is a fixed constant for the geometric intersection graph of several families of geometric objects like translated and rotated copies of a convex object in $\mathbb{R}^2$. However, the use of this parameter is not new. For example, in the offline setup, Marathe et al. [20] obtained a $2(\zeta - 1)$-approximation algorithm for the MCDS problem for any graph having an independent kissing number at most ζ.

Two geometric objects are said to be *non-overlapping* if they have no common interior, whereas we call them *non-touching* if their intersection is empty. An equivalent definition of independent kissing number ζ for a family $\mathcal{S}$ of geometric objects in $\mathbb{R}^d$ is given below.

Definition 2. *Let $\mathcal{S}$ be a family of geometric objects, and let u be any object belonging to the family $\mathcal{S}$. Let ζ_u be the maximum number of pairwise non-touching objects in $\mathcal{S}$ that we can arrange in and around u such that all of them are intersected by u. The independent kissing number ζ of $\mathcal{S}$ is defined to be* $\max_{u \in \mathcal{S}} \zeta_u$.

A set K of objects belonging to the family $\mathcal{S}$ is said to form an *independent kissing configuration* if (i) there exists an object $u \in K$ that intersects all objects in $K \setminus \{u\}$, and (ii) all objects in $K \setminus \{u\}$ are mutually non-touching to each other.

Here u and $K \setminus \{u\}$ are said to be the *core* and *independent set*, respectively, of the independent kissing configuration. The configuration is considered *optimal* if $|K \setminus \{u\}| = \zeta$, where ζ is the independent kissing number of $\mathcal{S}$. The configuration is said to be *standard* if all objects in $K \setminus \{u\}$ are mutually non-overlapping with u, i.e., their common interior is empty but touches the boundary of u.

A number of different definitions of fatness (not extremely long and skinny) are available in the geometry literature. For our purpose, we define the following. Let σ be an object and x be any point in σ. Let $\alpha(x)$ be the ratio between the minimum and maximum distance (under Euclidean norm) from x to the boundary $\partial(\sigma)$ of the object σ. In other words, $\alpha(x) = \frac{\min_{y \in \partial(\sigma)} d(x,y)}{\max_{y \in \partial(\sigma)} d(x,y)}$, where $d(.,.)$ denotes the Euclidean distance. The *aspect ratio* α of an object σ is defined as the maximum value of $\alpha(x)$ for any point $x \in \sigma$, i.e., $\alpha = \max\{\alpha(x) : x \in \sigma\}$. An object is said to be α*-fat object* if its aspect ratio is α. Observe that α-fat objects are invariant under translation, rotation and scaling. The *aspect point* of σ is a point in σ where the aspect ratio of σ is attained. The minimum distance (respectively, maximum distance) from the aspect point to the boundary of the object is referred to as the *width* (respectively, *height*) of the object. Note that fat objects are invariant under translation, rotation and scaling. For more details on α-fat object, one may see [9].

1.2 Related Work

The dominating set and its variants are well-studied in the offline setup. Finding MDS is known to be NP-hard even for the unit disk graphs [8,16]. A polynomial-time approximation scheme (PTAS) is known when all objects are homothets of a convex object [10]. King and Tzeng [18] initiated the study of the online MDS problem in Classical-Online-Model. They showed that for a general graph, the greedy algorithm achieves a competitive ratio of $n-1$, which is also a tight bound achievable by any online algorithm for the MDS problem, where n is the length of the input sequence. Even for the interval graph, the lower bound of the competitive ratio is $n-1$ [18]. Eidenbenz [13] proved that the greedy algorithm achieves a tight bound of 5 for the MDS of the unit disk graph.

Boyar et al. [3] considered a variant of the Relaxed-Online-Model for MDS, MIDS and MCDS problem in which, in addition to the Relaxed-Online-Model the revealed graph should always be connected. In this setup, they studied these problems for specific graph classes such as trees, bipartite graphs, bounded degree graphs, and planar graphs. Their results are summarized in [3, Table 2]. They proposed a 3-competitive algorithm for the MDS problem in the above-mentioned model for a tree. Later, Kobayashi [19] proved that 3 is also the lower bound for the tree in this setting. In the same setup, Eidenbenz [13] showed that, for the MCDS problem of unit disk graph, the greedy algorithm achieves a competitive ratio of $8+\epsilon$, whereas no online algorithm can guarantee a strictly better competitive ratio than 10/3. We observe that the (asymptotic) competitive ratio of the MCDS problem for the unit disk graph could be improved to 6.798 (see Section 2.2).

The minimum coloring problem is known to be NP-hard, even for unit disk graphs [8]. In the offline coloring case, a 3-approximation algorithm for coloring unit disk graphs was presented by Gräf et al. [17] and Marathe et al. [20]. Marathe et al. [20] generalised the approach for unit disk graphs to disk graphs. They proved that the obtained approximation for coloring disk graphs is at most 6. For online coloring, Erlebach and Fiala [14] proved that Algorithm-First-Fit achieves a competitive ratio of $O(\log n)$ for disk graphs. Recently, Albers and Schraink [1] proved that the best competitive ratio of both deterministic and randomized online algorithms for disk graphs is $\Theta(\log n)$. Capponi and Pilloto [7] proved that for any graph with an independent kissing number (see Defnition 1) at most ζ, popular Algorithm-First-Fit achieves a competitive ratio at most ζ. The existence of $O(\log m)$-competitive coloring algorithm for disk graphs, whose radius is in between $[1, m]$, is known due to Erlebach and Fiala [14]. In this paper, we generalize this result for the geometric intersection graph of bounded scaled α-fat objects in $\mathbb{R}^d$.

1.3 Our Contributions

In this paper, we obtain the following results.

1. First, we prove that for MDS and MIDS problems, the natural greedy algorithm has an optimal competitive ratio of ζ for a graph with an independent kissing number at most ζ (Theorem 1 and Theorem 2).
2. For the MCDS problem, we prove that, for any graph with the independent kissing number at most ζ, a greedy algorithm achieves a competitive ratio of at most 2ζ (Theorem 3). To complement this, we prove that the lower bound of the competitive ratio is at least $2(\zeta - 1)$ which holds even for a geometric intersection graph of translates of a convex object in $\mathbb{R}^2$ (Theorem 4).
3. Next, we consider coloring geometric intersection graphs of bounded scaled α-fat objects in $\mathbb{R}^d$ having a width in between $[1, m]$. For this, due to [7], the best known competitive ratio is ζ, where ζ is the independent kissing number of bounded scaled α-fat objects having a width in between $[1, m]$. Inspired by Erlebach and Fiala [14], we present Algorithm-Layer having a competitive ratio of at most $O(\zeta' \log m)$, where ζ' is the independent kissing number of bounded scaled α-fat objects having a width in between $[1, 2]$ (Theorem 5). Since the value of ζ could be very large compared to $\zeta' \log m$ (see Remark 2), it is a significant improvement.
4. All results obtained above for the MC problem, MDS problem and its variants depend on the graph parameter: the independent kissing number ζ. Therefore, the value of ζ becomes a crucial graph parameter to investigate. To estimate the value of ζ, we consider various families of geometric objects. We show that for congruent balls in $\mathbb{R}^3$ the value of ζ is 12. For translates of a regular k-gon ($k \in ([5, \infty) \cup \{3\}) \cap \mathbb{Z}$) in $\mathbb{R}^2$, we show that $5 \leq \zeta \leq 6$. While the value is 2^d for translates of a hypercube in $\mathbb{R}^d$, for congruent hypercubes in $\mathbb{R}^d$ the value is at least 2^{d+1}. We also give bounds on the value of ζ for α-fat objects in $\mathbb{R}^d$ having a width in between $[1, m]$. We feel that

these results will find applications in many problems. We illustrate a few in Section 5.

Note that all of our algorithms are deterministic. In particular, algorithms in items 1 and 2 do not need to know object's representation; whereas, ALGORITHM-LAYER needs to know the object's width upon arrival.

2 Dominating Set and its Variants

In this section, we discuss the well-known greedy online algorithms for MDS and its variants for graphs. We show how their performance depends on the independent kissing number ζ of the graph. Note that algorithms need not know the value of ζ in advance, and the object's representation is also unnecessary. All the missing proofs of this section will appear in the final version of the paper.

2.1 Minimum Dominating Set

The greedy algorithm, ALGORITHM-GDS, for finding a minimum dominating set is as follows. The algorithm maintains a feasible dominating set $\mathcal{A}$. Initially, $\mathcal{A} = \emptyset$. On receiving a new vertex v, if the vertex is not dominated by the existing dominating set $\mathcal{A}$, then update $\mathcal{A} \leftarrow \mathcal{A} \cup \{v\}$. Eidenbenz [13] showed that this algorithm achieves an optimal competitive ratio of 5 for the unit disk graph. It is easy to generalize this result for graphs with the fixed independent kissing number ζ.

Observation 1 *The vertices returned by the* ALGORITHM-GDS *are pairwise non-adjacent. In other words, the solution set is always an independent set.*

Theorem 1. ALGORITHM-GDS *has an optimal competitive ratio of ζ for the MDS problem of a graph having an independent kissing number at most ζ.*

As a result of Observation 1, the output produced by ALGORITHM-GDS is an independent dominating set. Thus, we have the following.

Theorem 2. ALGORITHM-GDS *has an optimal competitive ratio of ζ for the MIDS problem of a graph having an independent kissing number at most ζ.*

2.2 Minimum Connected Dominating Set

Recall that if we cannot add previously arrived vertices in the solution set for the minimum connected dominating set problem, the solution may result in a disconnected dominating set. Therefore, we use Relaxed-Online-Model for this problem. Here, in addition to the Relaxed-Online-Model, the revealed induced subgraph must always be connected for the MCDS problem. Eidenbenz [13] proposed a greedy algorithm for unit disk graph in the aforementioned setup and showed that the algorithm achieves a competitive ratio of at most $8 + \epsilon$.

We analyse the same algorithm for graphs with the fixed independent kissing number ζ.

Description of Algorithm-GCDS*:* Let V be the set of vertices presented to the algorithm and $\mathcal{A} \subseteq V$ be the set of vertices chosen by our algorithm such that $\mathcal{A}$ is a connected dominating set for the vertices in V. The algorithm maintains two disjoint sets $\mathcal{A}_1$ and $\mathcal{A}_2$ such that $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$. Initially, both $\mathcal{A}_1, \mathcal{A}_2 = \emptyset$. Let v be a new vertex presented to the algorithm. The algorithm first updates $V \leftarrow V \cup \{v\}$ and then does the following.

- If v is dominated by the set $\mathcal{A}$, do nothing.
- Otherwise, first, add v to $\mathcal{A}_1$. If v has at least one neighbour in V, choose any one neighbour, say u, of v from V, and add u to $\mathcal{A}_2$. In other words, update $\mathcal{A}_1 \leftarrow \mathcal{A}_1 \cup \{v\}$ and if necessary update $\mathcal{A}_2 \leftarrow \mathcal{A}_2 \cup \{u\}$. Note that u is already dominated by the existing dominating set $\mathcal{A}$. As a result, if we add u to the dominating set, it will result in a connected dominating set.

Note that the algorithm produces a feasible connected dominating set. The addition of vertex in $\mathcal{A}_1$ assures that $\mathcal{A}$ is a dominating set, and the addition of vertices in $\mathcal{A}_2$ ensures that $\mathcal{A}$ is a connected dominating set. Now, using induction, it is easy to prove the following.

Lemma 1. Algorithm-GCDS *maintains the following two invariants: (i)* $\mathcal{A}_1$ *is an independent set, and (ii)* $|\mathcal{A}_1| \geq |\mathcal{A}_2|$.

The next lemma is a generalization of a result by Wan et al. [23, Lemma 9].

Lemma 2. *Let* $\mathcal{I}$ *be an independent set, and* $\mathcal{O}$ *be a minimum connected dominating set of a graph with the independent kissing number* ζ*. Then* $|\mathcal{I}| \leq (\zeta - 1)|\mathcal{O}| + 1$.

Theorem 3. Algorithm-GCDS *has an asymptotic competitive ratio of at most* $2(\zeta - 1)$ *and an absolute competitive ratio of at most* 2ζ *for the MCDS problem for a graph having an independent kissing number at most* ζ*.*

Remark 1. Note that due to the result of Du and Du [11, Thm 1], for unit disk graphs, we have $|\mathcal{I}| \leq 3.399|\mathcal{O}| + 4.874$. As a result, similar to Theorem 3, one can prove that $|\mathcal{A}| \leq 6.798|\mathcal{O}| + 9.748$. Hence, for unit disk graphs, Algorithm-GCDS has an asymptotic competitive ratio of at most 6.798.

2.3 Lower Bound of the MCDS Problem

In this section, first, we propose a lower bound of the MCDS problem for a wheel graph. Then, using that, we propose a lower bound for the geometric intersection graph of translated copies of a convex object in $\mathbb{R}^2$.

Consider a *wheel graph* $W_k = (V, E)$ of order k, where $V = \{v_0, v_1, \ldots, v_k\}$ and $E = \big\{\{v_i, v_k\} \mid i \in [k-1]\big\} \cup \big\{\{v_i, v_{(i+1) \bmod k}\} \mid i \in [k-1]\big\}$. In other words, in W_k, the vertices $v_0, v_1, \ldots, v_{k-1}$ form a cycle C_k and a single core vertex v_k is adjacent to each vertex of C_k. Now, we define a cyclone-order of vertices in W_k.

(a) (b)

Fig. 1: (a) Optimal independent kissing configurations for unit disks; (b)Cyclone-order of vertices in a wheel graph. The bold arrow indicates a cyclone-order.

Definition 3 (Cyclone-order of vertices in a wheel graph). *For an integer* t $(0 < t < k-1)$, *in the the cyclone order of* W_k, *first, we enumerate* $t+1$ *vertices* $v_0, v_1, \ldots, v_t$ *of* C_k, *followed by an enumeration of the remaining* $k-t-1$ *vertices of* C_k, *i.e.,* $v_{k-1}, v_{k-2}, \ldots, v_{t+2}, v_{t+1}$. *Finally, the core vertex* v_k *is appended. We denote the first* $t+1$ *length sequence of* C_k *as a* cw-part *and the remaining* $k-t-1$ *length sequence as an* acw-part *of the cyclone-order.*

Now, it is easy to obtain the following lemma.

Lemma 3. *If the vertices of a wheel graph* W_k *are enumerated in a cyclone-order, then any deterministic online algorithm reports a CDS of size at least* $k-2$, *where the size of an offline optimum is 1.*

Now, we give an explicit construction of a wheel graph $W_{2\zeta}$ using translates of a convex object having independent kissing number ζ.

Lemma 4. *For a family of translates of a convex object having independent kissing number* ζ, *there exists a geometric intersection graph* $W_{2\zeta}$.

Combining Lemma 3 and Lemma 4, we have the following result.

Theorem 4. *Let* ζ *be the independent kissing number of a family* $\mathcal{S}$ *of translated copies of a convex object in* $\mathbb{R}^2$. *Then the competitive ratio of every deterministic online algorithm for MCDS of* $\mathcal{S}$ *is at least* $2(\zeta - 1)$.

3 Algorithm for the Minimum Coloring Problem

Here, we present Algorithm-Layer to find coloring for the geometric intersection graph of bounded scaled α-fat objects in $\mathbb{R}^d$ having a width in between $[1, m]$, where $m \geq 2$. First, we describe the well-known Algorithm-First-Fit as follows. Upon the arrival of the object σ, the algorithm assigns the smallest color available, i.e., the smallest color that has not yet been assigned to an adjacent vertex of σ.

Lemma 5. *[7, Lemma 4]* Algorithm-First-Fit *has a competitive ratio of* ζ *for the MC problem for a graph having an independent kissing number at most* ζ.

Now, we present a deterministic algorithm, Algorithm-Layer, that is similar to the algorithm of Erlebach and Fiala [14] originally defined for bounded scaled disks.

Description of Algorithm-Layer. For each $j \in \mathbb{Z}^+ \cup \{0\}$, let L_j be the jth layer containing all objects with a width in between $[2^j, 2^{j+1})$. Observe that the width of each layer's objects falls within a factor of two. For each layer L_j, we use Algorithm-First-Fit separately to color the objects. When an object σ_i having width w_i arrives, our algorithm, first, determines the layer number $j = \lfloor \log w_i \rfloor$. Then we color σ_i using Algorithm-First-Fit considering already arrived objects in L_j, and also we use the fact that a color that is used in any other layer cannot be used for σ_i. A pseudo-code of Algorithm-Layer is given in Algorithm 1.

Algorithm 1 Algorithm-Layer

$L_j \leftarrow \emptyset$, for all $j \in \mathbb{Z}^+$
for $i = 1$ to n; **do** ▷ arrival of an object σ_i having a width w_i
  begin
    $j = \lfloor \log_2 w_i \rfloor$; ▷ Identifying the index of the layer to which σ_i belongs, where r_i is the width of σ_i.
    $L_j \leftarrow L_j \cup \{\sigma_i\}$; ▷ The layer containing σ_i
    $F = \{c(\sigma_k) : 1 \leq k < i, \sigma_k \in L_j, \sigma_k \cap \sigma_i \neq \emptyset\} \cup \{c(\sigma_k) : 1 \leq k < i, \sigma_k \notin L_j\}$; ▷ The set of forbidden colors
    $c(\sigma_i) = \min\{\mathbb{Z}^+ \setminus F\}$; ▷ color assigned to σ_i
  end
end for

Theorem 5. *Let ζ' be the independent kissing number of bounded scaled α-fat objects having a width in between $[1, 2]$.* Algorithm-Layer *has a competitive ratio of at most $O(\zeta' \log m)$ for MC of geometric intersection graph of bounded scaled α-fat objects in $\mathbb{R}^d$ having width in between $[1, m]$.*

Proof. Let $\mathcal{A}$ and $\mathcal{O}$ be the set of colors used by the Algorithm-Layer and an offline optimum for an input sequence $\mathcal{I}$. For $i \in \{0, 1, \ldots, \lceil \log m \rceil\}$, let the layer L_i be the collection of all α-fat objects in $\mathcal{I}$ having a width in $\left[2^i, 2^{i+1}\right)$. Let $\mathcal{O}_i$ be a set of colors used by an offline optimum algorithm for the layer L_i. Let $\mathcal{O}'_i \subseteq \mathcal{O}$ be the set of colors used for the layer L_i. Note that the set of colors in $\mathcal{O}'_i$ is a valid coloring for objects in L_i. Thus, we have $|\mathcal{O}_i| \leq |\mathcal{O}'_i|$. Let $\mathcal{A}_i$ be the set of colors used by Algorithm-Layer to color layer L_i. Note that $\mathcal{A} = \cup_{i=0}^{\lceil \log m \rceil} \mathcal{A}_i$ and $\mathcal{A}_i \cap \mathcal{A}_j = \emptyset$, where $i \neq j \in [\lceil \log m \rceil]$. Due to Lemma 5, for all $i \in [\lceil \log m \rceil]$ we have $|\mathcal{A}_i| \leq \zeta_i |\mathcal{O}_i|$, where ζ_i is the independent kissing number of bounded scaled α-fat objects having width in between $[2^i, 2^{i+1})$. Since the width of objects in each layer is within a factor of two, for each i, the value of ζ_i is the same as ζ'. Since $|\mathcal{O}_i| \leq |\mathcal{O}'_i| \leq |\mathcal{O}|$, we have $|\mathcal{A}_i| \leq \zeta' |\mathcal{O}'_i| \leq \zeta' |\mathcal{O}|$. Then, we have $|\mathcal{A}| = \sum_{i=0}^{\lceil \log m \rceil} |\mathcal{A}_i| \leq \sum_{i=0}^{\lceil \log m \rceil} \zeta' |\mathcal{O}_i| = O(\zeta' \log m) |\mathcal{O}|$. Hence, the theorem follows. □

4 Value of ζ for Families of Geometric Objects

Note that the value of ζ for unit disk graphs is already known to be 5 [13]. Here, we study the value of ζ for other geometric intersection graphs.

Theorem 6. *The independent kissing number for the family of*

(a) congruent balls in $\mathbb{R}^3$ is 12;
(b) translated copies of a hypercube in $\mathbb{R}^d$ is 2^d, where $d \in \mathbb{Z}^+$;
(c) translated copies of an equilateral triangle is at least 5 and at most 6;
(d) translated copies of a regular k-gon ($k \geq 5$) is at least 5 and at most 6;
(e) congruent hypercubes in $\mathbb{R}^d$ is at least 2^{d+1}, where $d \geq 2$ is an integer;
(f) bounded-scaled α-fat objects having a width in between $[1, m]$ is at least $\left(\frac{\alpha}{2}\left(\frac{m+2}{1+\epsilon}\right)\right)^d$ and at most $\left(\frac{m}{\alpha}+2\right)^d$, where $\epsilon > 0$ is a very small constant.

For each item (except (e)) of the above theorem, we prove both the upper and lower bounds of the value of the independent kissing number.

Proof of Theorem 6(a).

First, we present a lower bound. Let C be a regular icosahedron whose each edge is of length ℓ=$2+\epsilon$, where $0 < \epsilon < 1$ (in particular, one can choose ϵ=0.001). Let corner points of C be the centers of unit (radius) balls $\sigma_1, \sigma_2, \ldots, \sigma_{12}$. Since the edge length of the icosahedron C is greater than 2, all these balls are mutually non-touching. Let B be the circumscribed ball of the icosahedron C. It is a well-known fact that if a regular icosahedron has edge length ℓ, then the radius r of the circumscribed ball is $r = \ell \sin(\frac{2\pi}{5})$ [21]. In our case, it is easy to see that $r < 2$. In other words, the distance from the center of B to each corner point of C is less than two units. Thus, each of these unit balls $\sigma_1, \sigma_2, \ldots, \sigma_{12}$ is intersected by a unit ball σ_{13} whose center coincides with the center of B. This implies that the value of ζ for congruent balls in $\mathbb{R}^3$ is at least 12. The upper bound follows from the fact that the independent kissing number for balls in $\mathbb{R}^3$ is 12 [5,22]. Hence, the theorem follows. □

Proof of Theorem 6(b).

Upper Bound Let K be an optimal independent kissing configuration for translates of an axis-parallel unit hypercube in $\mathbb{R}^d$. Let the core of the configuration be u. It is easy to observe that an axis-parallel hypercube R, with a side length of 2 units, contains all the centers of hypercubes in $K \setminus \{u\}$. Let us partition R into 2^d smaller axis-parallel hypercubes, each having unit side length. Note that each of these smaller hypercubes can contain at most one center of a hypercube in $K \setminus \{u\}$. As a result, we have $|K \setminus \{u\}| \leq 2^d$. Therefore, the independent kissing number for translates of a hypercube in $\mathbb{R}^d$ is at most 2^d.

Lower Bound We give an explicit construction of an independent kissing configuration K where the size of the independent set is 2^d. Let $\sigma_1, \sigma_2, \ldots, \sigma_{2^d}$ and σ_{2^d+1} be the d-dimensional axis-parallel unit hypercubes of K. We use c_i to denote the center of σ_i, for $i \in [2^d+1]$. Let the center $c_{2^d+1} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, and $p_1, p_2, \ldots, p_{2^d} \in \mathbb{R}^d$ be corner points of the hypercube σ_{2^d+1}. It is easy to observe that each coordinate of $p_i, i \in [2^d]$ is either 0 or 1. Let ϵ be a positive constant satisfying $0 < \epsilon < \frac{1}{2\sqrt{d}}$. For $i \in [2^d]$ and $j \in [d]$, let us define the jth coordinates of c_i as follows:

$$c_i(x_j) = \begin{cases} -\epsilon, & \text{if } p_i(x_j) = 0 \\ 1+\epsilon, & \text{if } p_i(x_j) = 1, \end{cases} \tag{1}$$

where $c_i(x_j)$ and $p_i(x_j)$ are the j^{th} coordinate value of c_i and p_i, respectively.

To complete the proof, here, we argue that the hypercubes $\sigma_1, \sigma_2, \ldots, \sigma_{2^d}$ are mutually non-touching, and each intersected by the hypercube σ_{2^d+1}. To see this, first, note that, for any $i \in [2^d]$, the Euclidean distance $d(p_i, c_i)$ between p_i and c_i is $\sqrt{d}\epsilon$ (follows from Equation 1). Since $\epsilon < \frac{1}{2\sqrt{d}}$, we have $d(p_i, c_i) < \frac{1}{2}$. As a result, the corner point p_i of σ_{2^d+1} is contained in the hypercube σ_i. Thus, σ_{2^d+1} intersects σ_i, $\forall\ i \in [2^d]$. Now, consider any $i, j \in [2^d]$ such that $i \neq j$. Since p_i and p_j are distinct, it is easy to note that they will differ in at least one coordinate. As a result, the distance between c_i and c_j is $(1+2\epsilon)$ under L_∞-norm. So, σ_i and σ_j are non-touching. Hence, the theorem follows. □

The proof of the remaining parts of Theorem 6 will appear in the final version of the paper.

Remark 2. Let ζ and ζ' be the independent kissing number for the family of bounded-scaled α-fat objects having a width in between $[1, m]$ and $[1, 2]$, respectively. Now, we have the following remarks.

(i) $\zeta' \leq \left(\frac{2}{\alpha} + 2\right)^d$: It follows from Theorem 6(f) by putting the value of $m = 2$.

(ii) $\zeta \geq \left(\frac{(m+2)\alpha^2}{4(1+\epsilon)(1+\alpha)}\right)^d \zeta'$: Due to Theorem 6(f), we have $\zeta \geq \left(\frac{\alpha}{2}\left(\frac{m+2}{1+\epsilon}\right)\right)^d$. Note that $\zeta \geq \left(\frac{\alpha}{2}\left(\frac{m+2}{1+\epsilon}\right)\right)^d = \left(\frac{\alpha}{2}\left(\frac{m+2}{1+\epsilon}\right)\right)^d \left(\frac{1}{\frac{2}{\alpha}+2}\right)^d \left(\frac{2}{\alpha}+2\right)^d$. Now, using the fact $\zeta' \leq \left(\frac{2}{\alpha}+2\right)^d$ in the above expression, we have $\zeta \geq \left(\frac{(m+2)\alpha^2}{4(1+\epsilon)(1+\alpha)}\right)^d \zeta'$.

5 Applications

In this section, we mention some of the implications of Theorem 6. Combining Theorem 6 with Theorem 1, Theorem 2 and Lemma 5, respectively, we obtain the following results for the online MDS, MIDS and MC problems, respectively.

Theorem 7. *For each of the MDS, MIDS and MC problems, there exists a deterministic online algorithm that achieves a competitive ratio of*

(a) 12 for congruent balls in $\mathbb{R}^3$*;*
(b) 2^d *for translated copies of a hypercube in* $\mathbb{R}^d$*, where* $d \in \mathbb{Z}^+$*;*
(c) at most 6 *for translated copies of a regular* k*-gon (for* $k = 3$ *and* $k \geq 5$*);*
(d) at most $\left(\frac{m}{\alpha} + 2\right)^d$ *for bounded-scaled* α*-fat objects having width in between* $[1, m]$.

Similarly, combining Theorem 3 and Theorem 6, we have the following.

Theorem 8. *For the MCDS problems, there exists a deterministic online algorithm that achieves a competitive ratio of*

(a) *22 for congruent balls in* $\mathbb{R}^3$*;*
(b) $2(2^d - 1)$ *for translated copies of a hypercube in* $\mathbb{R}^d$*, where* $d \in \mathbb{Z}^+$*;*
(c) *at most* 10 *for translated copies of a regular* k*-gon (for* $k = 3$ *and* $k \geq 5$*);*
(d) *at most* $2\left(\left(\frac{m}{\alpha}+2\right)^d - 1\right)$ *for bounded-scaled* α*-fat objects having width in* $[1, m]$.

Now, we present the implication of Theorem 6 on the online t-relaxed coloring problem. For a nonnegative integer t, in the *online t-relaxed coloring* problem, upon the arrival of a new vertex, the algorithm must immediately assign a color to it, ensuring that the maximum degree of the subgraph induced by the vertices of this color class does not exceed t. The objective of the problem is to minimize the number of distinct colors. Combining the result of Capponi and Pilotto [7, Thm 5] with Theorem 6, we have the following.

Theorem 9. *For online* t*-relaxed coloring problem, there exists an online algorithm that achieves an asymptotic competitive ratio of*

(a) *288 for congruent balls in* $\mathbb{R}^3$*;*
(b) 2^{2d+1} *for translated copies of a hypercube in* $\mathbb{R}^d$*, where* $d \in \mathbb{Z}^+$*;*
(c) *at most* 72 *for translated copies of a regular* k*-gon (for* $k = 3$ *and* $k \geq 5$*);*
(d) *at most* $2\left(\frac{m}{\alpha}+2\right)^{2d}$ *for bounded-scaled* α*-fat objects having width in between* $[1, m]$.

6 Conclusion

We conclude by mentioning some open problems. The results obtained in this paper, as well as the results obtained in [7,20] for other graph problems, are dependent on the independent kissing number ζ. Consequently, the value of ζ becomes an intriguing graph parameter to investigate. For congruent balls in $\mathbb{R}^3$ and translates of a hypercube in $\mathbb{R}^d$, we prove that the value of ζ is tight and equals 12 and 2^d, respectively. In contrast, the value of ζ for translates of a regular k-gon (for $k \in ([5,\infty) \cup \{3\}) \cap \mathbb{Z}$) is either 5 or 6. We propose to settle the value ζ for this case as an open problem. For congruent hypercubes in $\mathbb{R}^d$, we prove that the value of ζ is at least 2^{d+1}; on the other hand, since congruent hypercubes are $\frac{1}{\sqrt{d}}$-fat objects, due to Theorem 6(f), it follows that ζ is at most $(2+\sqrt{d})^d$. Bridging this gap would be an open question. It would be of independent interest to see parametrized algorithms for graphs considering ζ as a parameter.

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