Guarding Polyhedral Terrain by *k*-Watchtowers

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Abstract. The discrete k-watchtower problem for a polyhedral terrain $\mathcal{T}$ with n vertices is to find k vertical segments, called watchtowers, of smallest common height, whose bottom end-points (bases) lie on some vertices of $\mathcal{T}$, and every point of $\mathcal{T}$ is visible from the top end-point of at least one of those vertical segments. Agarwal et al. [1] proposed a polynomial time algorithm using parametric search technique for this problem with $k = 2$. Surprisingly, no result is known for the problem when $k > 2$. In this paper, we propose an easy to implement algorithm to solve k-watchtower problem in $\mathbb{R}^3$ for a fixed constant k. Our algorithm does not use parametric search.

Keywords - Watchtower problem, polyhedral terrains, visibility.

1 Introduction

A polyhedral terrain in $\mathbb{R}^3$ is a connected 3D polyhedral surface such that for each point $v = (x, y, z)$ on the surface, $z = g(x, y)$ for some linear function g [9]. In other words, any vertical line intersects a terrain at most once and the orthogonal projection of a terrain on the XY-plane is a (bounded) planar subdivision. In general, a polyhedral terrain in $\mathbb{R}^d$ is the graph of a continuous, piecewise-linear $(d-1)$- variate function [1].

The problem of placing watchtowers on a polyhedral terrain is a matter of great interest due to its application in surveillance, navigation, computer vision, modelling and graphics, geographic information system, etc.. Here the objective is to place a given (k) number of watchtowers on the vertices of a polyhedral terrain such that every point in the surface of the terrain is visible from at least one of the watchtowers and the maximum height among these watchtowers is minimized. This is also known as *discrete k-watchtower problem* where base of each watchtower is restricted to the vertices of the terrain. In the *continuous* version, base of watchtower can be placed anywhere in the terrain.

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Cole and Sharir [3] showed that the problem of finding minimum number of guards, where guards are to be placed on the terrain without any elevation, to guard the terrain is NP-hard. Sharir [7] proposed polynomial time algorithm for the continuous one watchtower placement problem for polyhedral terrain in $\mathbb{R}^3$ that runs in $O(n \log^2 n)$ time. Later the time complexity of the problem was improved to $O(n \log n)$ by Zhu [10]. Agarwal et al. [1] proposed a deterministic polynomial time algorithm for the discrete two watchtower problem for polyhedral terrain in $\mathbb{R}^3$. The time complexity of their algorithm is $O(n^{11/3} polylog(n))$, and it uses the parametric search technique of Meggido [5]; here n is the number of vertices of the polyhedral terrain. Agarwal et al. [1] mentioned the k-watchtower problem for $k > 2$ as open. For the continuous version of the problem in $\mathbb{R}^3$, no result is known for $k > 1$.

In this paper, we propose a general polynomial time algorithm for discrete k-watchtower problem for any fixed integer $k > 2$. In Section 3, we first develop an algorithm to decide whether it is feasible to guard the entire terrain using k-watchtowers for a given height h, that runs in $O(n^{k+3} k^2 \alpha^2(n) \log n)$ time. We use this decision procedure to do a binary search over the height to find the optimum height. As the domain of height is continuous, we need to discretise it. In Section 4, we describe the details of this discretisation. The running time of our algorithm is $O(n^{k+3} k^2 \alpha^2(n) \log^2 n + n^7 \alpha^3(n) \log n)$. The strength of our algorithm is that it is simpler to implement as we do not use parametric search.

2 Preliminaries

We use $[k]$ as a shorthand notation of $\{1, 2, \dots, k\}$. Without loss of generality, we assume that each facet of the polyhedral terrain is a triangle. In other words, a terrain in $\mathbb{R}^3$ is denoted by $\mathcal{T}(V, E, F)$, where V, E and F denote the set of vertices, edges and facets of $\mathcal{T}$, respectively. The boundary of a facet $f \in F$ is denoted by δf.

A watchtower is a vertical line segment whose bottom end point/base lies on a vertex of $\mathcal{T}$. We use $u(h)$ to denote a watchtower based at a vertex $u \in \mathcal{T}$ and at a height h from u. A point $p \in \mathcal{T}$ is said to be *visible* from the watchtower $u(h)$ if the line segment $\overline{p, u(h)}$ lies fully above the terrain.

We say that k watchtowers $u_1(h), u_2(h), \dots, u_k(h)$ *guard* the entire terrain $\mathcal{T}$ if each point on the terrain $\mathcal{T}$ is visible from at least one of those k watchtowers; we refer to $(u_1, u_2, \dots, u_k)$ as a *guard k-tuple* at height h.

The *invisibility region* $H_{u(h)}(f)$ consists of all the points on the facet f that are not visible from $u(h)$. The region $f \setminus H_{u(h)}(f)$ is referred to as *visibility region* of $u(h)$ in the facet f. Note that if $H_{u(h)}(f) \neq \emptyset$, then $H_{u(h)}(f)$ is a collection of *invisibility polygons.* We denote each vertex and edge of an invisibility polygon as *breakpoint* and *invisibility segment*, respectively. Similarly, $H_{u(h)}(e)$ consists of all the points on the edge $e \in E$ that are not visible from $u(h)$. Note that if $H_{u(h)}(e) \neq \emptyset$, then $H_{u(h)}(e)$ is a collection of *invisibility intervals* on the edge e. We use $H_{u(h)}$ to denote all the points of $\mathcal{T}$ that are not visible from $u(h)$.

Computing $H_{u(h)}(f)$: Observe that boundary of $H_{u(h)}(f)$ is formed by shadows of some edges of $\mathcal{T}$ on the facet f assuming a light source at $u(h)$. As noted by Agarwal et al. [1, §5], $H_{u(h)}(f)$ is the upper envelope of $O(n)$ line segments, where each line segment is generated due to intersection of facet f with another plane defined by $u(h)$ and an edge in $\mathcal{E}$. Here $\mathcal{E} = E \cup \{\text{line segment } \overline{v, v'} | v \in V\}$, where v' is a projection of v on the XY plane (assuming that all the points of V are above the XY plane). Thus, the combinatorial complexity of $H_{u(h)}(f)$ is $O(n\alpha(n))$, where α is inverse Ackerman function [8]. Due to the result of Hersberger [4], we can compute the region $H_{u(h)}(f)$ in $O(n \log n)$ time.

It is easy to see that for an edge e that is shared by two facets f_1 and f_2, we can obtain the sorted list of invisibility intervals of $H_{u(h)}(e)$ after computing both $H_{u(h)}(f_1)$ and $H_{u(h)}(f_2)$, and spending $O(n \log n)$ time.

However, Agarwal et al. [1, §5] mentioned that considering all the facets, the combinatorial complexity of the region $H_{u(h)}$ is $O(n^2)$, and we can also compute the region $H_{u(h)}$ in nearly quadratic time.

3 The decision procedure

The decision version of the problem is as follows: *Given a polyhedral terrain $\mathcal{T}$, height $h \in \mathbb{R}$ and a constant $k \in \mathbb{N}$, decide whether there exists a guard k-tuple at height h for the terrain $\mathcal{T}$.*

Lemma 1 *A k-tuple $(u_1, u_2, \ldots, u_k) \in V^k$ is a guard k-tuple at height h for the terrain $\mathcal{T}$ if and only if the following two properties are satisfied:*

(i) Each edge is collectively visible from k watchtowers $u_1(h), u_2(h), \ldots, u_k(h)$ *i.e. there exist no edge $e \in E$ such that there is a common point in the intersection of k invisibility intervals on e due to these k watchtowers. Formally, $\cap_{i=1}^{k} H_{u_i(h)}(e) = \phi$, $\forall e \in E$.*

(ii) Each facet is collectively visible from k watchtowers $u_1(h), u_2(h), \ldots, u_k(h)$ *i.e. there does not exist a facet f such that there is a common point in the intersection of k invisibility polygons on f due to these k watchtowers. Formally, $\cap_{i=1}^{k} H_{u_i(h)}(f) = \phi$, $\forall f \in F$.*

Proof. The necessity of the above statement is trivially true. Let's consider sufficiency condition i.e. if above two properties are satisfied then the entire terrain $\mathcal{T}(V, E, F)$ is visible from the given k watchtowers. The property (ii) ensures that each facet $f \in F$ is visible from at least one of the k watchtowers. However, it is not sufficient to guard the entire terrain as there may exist some part of edge or some vertices which are not visible by any of the watchtowers. For example, in Fig. 1 for two watchtowers; the left portion the shared edge between two facets is not visible by either of the two watchtowers, though both the facets are visible. This situation can be generalized for k watchtowers as well. The property (i)

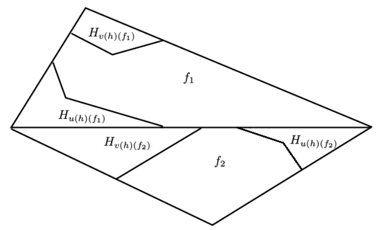

Fig. 1. Facets f_1 and f_2 are visible but part of the shared edge e is not visible from both $u(h)$ and $v(h)$

ensures that each edge and vertex of the terrain is visible. Therefore, satisfying these two properties imply that all the elements in set V, E and F of $\mathcal{T}$ are visible from the said k watchtowers. □

The decision procedure proceeds in two steps. In the first step, it discards all k-tuples $(u_1, u_2, \ldots, u_k) \in V^k$ that do not satisfy the property (i) of the Lemma 1. Considering the remaining k-tuples, in the second step, it rejects each of those k-tuples that violates property (ii) of the Lemma.

First Step: Given an edge e and k-watchtowers of height h at $u_1, u_2, \ldots, u_k \in V$, we can compute $H_{u_i(h)}(e)_{end}$ $\forall i \in [k]$ as mentioned in Section 2 in $O(nk \log n)$ time. Note that in each list, the end-points are already in sorted order. To check the existence of a point $p \in e$ such that $p \in \cap_{i=1}^{k} H_{u_i(h)}(e)$, we need to merge these sorted k lists. Using a min-heap of size k, we can do this in $O(nk \log k)$ time. If such a point $p \in e$ exists, then e is not completely visible by the k-watchtowers $\{u_1(h), u_2(h), \ldots, u_k(h)\}$. For a given k-tuple, we check this observation for all edges $e \in E$ in $O(n^2 k \log n)$ time, and reject the tuple if we find an edge which is not guarded. Since we need to repeat this process for all possible k-tuples in V, the overall time complexity of this step is $O(n^{k+2} k \log n)$.

Second Step: For a given facet f and k watchtowers of height h at $u_1, u_2, \ldots, u_k \in V$, first we compute $H_{u_i(h)}(f)$ for all $i \in [k]$. Now, we need to test whether $\cap_{i=1}^{k} H_{u_i(h)}(f) = \emptyset$ or not.

Let $S_i(f)$ be the set of all edges of $H_{u_i}(f)$, and $S(f) = \cup_{i=1}^{k} S_i(f)$. As number of line segments in $S_i(f)$ is $O(n\alpha(n))$, we have $|S(f)| = O(nk\alpha(n))$, where $|A|$ denotes the number of elements in set A. Let $I_{S(f)}$ be the set of points generated by pairwise intersection of the line segments in $S(f)$. Here, $|I_{S(f)}| = O(n^2 k^2 \alpha^2(n))$. The necessary and sufficient condition that k watchtowers can guard a given facet f is as follows.

Lemma 2 *A facet $f \in F$ is guarded by $u_1(h), \ldots, u_k(h)$ if and only if there does not exist any point $p \in I_{S(f)}$ satisfying the following:*

– *p is inside the invisibility regions (boundary included) $H_{u_i(h)}(f)$ $\forall$ $i \in [k]$.*

Proof. The necessity of above statement is trivially true. So, let us consider the sufficiency condition i.e if the above condition is true then the facet f is completely visible from k watchtowers. For a contradiction, let us assume that there exists a point p' at the facet f which is not visible by any of the k watchtowers. This means that this point p' is inside or on the boundary of common intersection region R of $H_{u_i(h)}(f)$, $i \in [k]$. The extreme points of this closed region R are in the set $I_{S(f)}$ and the proof follows. □

For each $i \in [k]$, we separately create a data structure D_i by triangulating the interior of the polygons in $H_{u_i(h)}(f)$ and also the exterior of $H_{u_i(h)}(f)$. In D_i, each interior and exterior triangles are given weight 1 and 0, respectively. Since the combinatorial complexity of each $H_{u_i(h)}(f)$ is $O(n\alpha(n))$, the size of each of these data structures is $O(n\alpha(n))$, and can be created in $O(n\alpha(n)\log n)$ time, and point location can be performed in $O(\log n)$ time [2].

Now, for each point $p \in I_{S(f)}$, we search p in all the data structures $D_i, i \in [k]$. In each D_i, we identify the triangle in which p lies, and add its weight in a counter $\chi(p)$. If $\chi(p) = k$, p is not guarded. Thus, for each point $p \in I_{S(f)}$, we spend $O(k \log n)$ time to decide whether q is inside common invisibility region $\cap_i^k H_{u_i(h)}(f)$. We need to do this test for all the points in $\cup_{f\in F} I_{S(f)}$. Since $|I_{S(f)}| = O(n^2k^2\alpha^2(n))$, the total number of intersection points in $\cup_{f\in F} I_{S(f)}$ is $O(n^3k^2\alpha^2(n))$.

Thus, Step 2 takes total $O(n^{k+3}k^2\alpha^2(n)\log n)$ time, and we have the following.

Lemma 3 *Given a height $h \in \mathbb{R}$ and a polyhedral terrain $\mathcal{T}$, we can decide whether there exists k-watchtowers of height h that can guard the terrain $\mathcal{T}$ in $O(n^{k+3}k^2\alpha^2(n)\log n)$ time. We can also report all possible sets of k watchtowers of height h that can guard the terrain within same amount of time.*

4 Algorithm via discretisation of height

Given a polyhedral terrain $\mathcal{T}$, in this section, we discuss the algorithm for finding the minimum height h^* for which we can find k-watchtowers of height h^* that can guard $\mathcal{T}$.

Overview of the Algorithm Our algorithm consists of two phases. In each phase, we enumerate a set of discrete heights, and perform binary search on them using the decision procedure described in the previous section.

In the first phase, we consider each vertex $u \in V$ as a possible location of a watchtower. Starting from $h = 0$, if we increase the height of the watchtower continuously until $u(h)$ sees the whole terrain, the invisibility region $H_{u(h)}(f)$ on each facet $f \in F$ shrinks continuously. During this process, the shadow of an edge appears or disappear at some specific heights, called *critical heights.* Considering each vertex as a possible location of a watchtower, we enumerate all the critical heights. The number of such critical heights is $O(n^4)$, where n is the number of vertices of the watchtower. We perform binary search among the sorted list $L_{critical}$ of these critical heights. For each choice of critical height, we test whether this height is feasible for guarding $\mathcal{T}$ by any tuple of k watchtowers (see Section 3). Thus, we get a pair of consecutive heights $h', h'' \in L_{critical}$ such that h' is not feasible but h'' is feasible, and the optimum height $h^* \in (h', h'']$.

Note that we have at least one k-tuple $(u_i, u_2, \ldots, u_k) \in V^k$ such that each point on $\mathcal{T}$ is visible by at least one of the watchtowers $\{u_i(h'')|i \in [k]\}$, and there is no common intersection region of the invisible regions $H_{u_i(h'')}$, $i \in [k]$. We may further decrease the height from h'' as long as there is no point $p \in \mathcal{T}$ that is invisible from all of the watchtowers. This event will occur when all of the invisible regions $H_{u_i(h)}$ has a common intersection point. In this situation, we can not decrease the height of watchtowers corresponding to the k-tuple $(u_i, u_2, \ldots, u_k)$ to guard the whole terrain. In the second phase, we enumerate all possible (contact) heights where this type of events might occur considering all possible positions of watchtowers. As in the first phase, we sort this set of contact heights to obtain a sorted list $L_{contact}$ of (contact) heights. Finally, to obtain the optimum h^*, we perform a binary search using the decision procedure discussed in Section 3.

4.1 Phase 1: Critical-height generation

Recall that a typical invisibility region $H_{u(h)}(f)$ is a collection of simple invisibility polygons. For each invisibility polygon, at least one invisibility segment is a portion of the boundary of f. Now, let us consider the breakpoints of $H_{u(h)}(f)$ that are not the end-points of δf. These breakpoints can, in general, be categorized as follows:

Category-1: This type of breakpoint is generated by the projection of each vertex of $\mathcal{T}$ on the facet f.

Category-2: This type of breakpoint is generated by the intersection of the projections of each pair of edges in $E \setminus \{\text{edges of } \delta f\}$ of $\mathcal{T}$ on the facet f.

Category-3: This type of breakpoint is generated by the intersection of the projection of edge $e \in E \setminus \{\text{edges of } \delta f\}$ of $\mathcal{T}$ on the facet f and an edge of the facet f.

Thus, each breakpoint of $H_{u(h)}(f)$ is associated with either a vertex $v \in \mathcal{T}$, or two edges from $E \setminus \delta f$, or one edge from δf and another from $E \setminus \delta f$.

Initially, let's consider that height $h = 0$, and a watchtower is based at $u \in \mathcal{T}$. Now, we increase the height continuously until $u(h)$ sees the entire terrain. During this process, we observe some distinct heights where the topology (set of breakpoints) of $H_{u(h)}(f)$ changes. We refer to such a height as a *critical height*. We can classify the events corresponding to these as follows.

Type-A Event: When a breakpoint p of Category-1 appears/disappears from the facet f at some height h, we call this event as Type-A event.

Type-B Event: When two breakpoint p, q of Category-2 merge to form a single breakpoint of $H_{u(h)}(f)$; we call such an event as Type-B Event. This happens

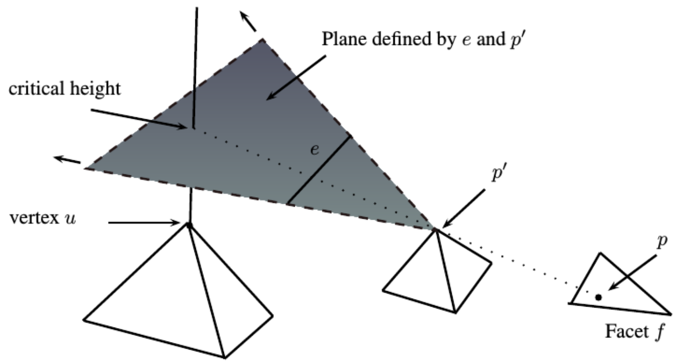

Fig. 2. Computing critical heights for Type-A Events

when two breakpoints that are consecutive on the boundary of invisibility region (but not on δf) merge to form a single vertex.

Type-C Event: When two breakpoint p, q, such that at least one of them is Category-3, merge to form a single breakpoint of $H_{u(h)}(f)$, we call such an event as Type-C Event.

It is easy to see that no other event can change the topology of $H_{u(h)}(f)$. Considering all the facets $f \in F$ and all positions $u \in V$, first we analyze the Type-A events; next we consider Type-B and Type-C events together.

Type-A events: Let p be a breakpoint of Category-1. It is projection of a vertex $p' \in V$ due to a watchtower based at a vertex $u \in V$. Assume that we are continuously increasing the height of the watchtower based at u, and the breakpoint p appears at some critical height h. This happens when the tip of the watchtower $u(h)$, the vertex $p' \in V$ and some edge $e \in E$ become coplanar (see Fig. 2). Consider the plane defined by p' and e. Note that this plane intersects the vertical line passing through u at a height h.

Let $adj(u)$ be the set of edges in E that are adjacent to u, i.e., $adj(u) = \{uy | uy \in E, y \in V \setminus \{u\}\}$. Each vertex-edge pair (v, e), $v \in V \setminus \{u\}$ & $e \in E \setminus adj(u)$ defines a plane. Thus, we have $O(n^2)$ such planes. Intersection of each plane with the vertical line at $u \in V$ gives a critical height at u for Type-A event. Considering all the vertices $u \in V$, we have $O(n^3)$ critical heights corresponding to all Type-A event, and all of these can be enumerated in $O(n^3)$ time.

Type-B and Type-C events: We first consider the Type-B events. Let p_1, p_2 be two Category-2 breakpoints that are consecutive on the boundary of $H_{u(h)}(f)$. Let e_1, e_2 be the edges of $H_{u(h)}(f)$ incident to p_1, and e_2, e_3 be the edges of $H_{u(h)}(f)$ incident to p_2. Let e_1, e_2, e_3 are projections of edges $e'_1, e'_2, e'_3 \in E$, respectively. Assume that we are continuously increasing the height of the watchtower based at u; at some critical height h', we notice that p, q merge to form a single breakpoint. Clearly, this event happens when the projections of e'_1, e'_2 and e'_3 intersect at a common point on f. In other words, this happens for a view-point from where the edges e'_1, e'_2 and e'_3 appear to intersect at a single point. Plantinga and Dyer [6] showed that this viewing direction is :

$d = [(p_{11}+t(p_{11}-p_{12})-p_{21}) \times (p_{21}-p_{22})] \times [(p_{11}+t(p_{11}-p_{12}-p_{21}) \times (p_{21}-p_{22})]$,

where p_{i1}, p_{i2} are the two end points of the edge e'_i, $i \in [3]$, and $0 \leq t \leq 1$. Note that d is given as Cartesian coordinates, and it is a quadratic function on t. Thus, for a given vertex $v = (x, y, z) \in V$, we can find the corresponding critical heights due to a triple of edges e'_1, e'_2 and e'_3 in $O(1)$ time.

Assuming a vertex $u \in V$ as a possible position of watchtower, we need to consider each triple of edges $e'_1, e'_2, e'_3 \in E$, define the corresponding viewing direction d and compute its intersection with the vertical line at $u \in V$. Thus, the number of Type-B events for a vertex $u \in V$ is $O(n^3)$. Considering all the vertices in V, we have in total $O(n^4)$ Type-B events.

Note that a Type-C event corresponds to the event when at least one of the edges from e_1, e_2 and e_3 of $H_{u(h)}(f)$ is a part of the facet f. Thus, this event corresponds to the viewing direction from where e'_1, e'_2 and e'_3 appear to intersect at a single point, and one of the edges is on the boundary of the facet f. It is easy to note that during the process of enumerating all critical heights corresponding to the Type-B events, we also enumerated critical heights corresponding to Type-C events.

Let $L_{critical}$ be the sorted list of the union of all the critical heights generated by Type-A, Type-B and Type-C events considering all the vertices $u \in V$ together. It is easy to see the following.

Lemma 4 *Given a polyhedral terrain $\mathcal{T}$ with n vertices, the size of the sorted list $L_{critical}$ is $O(n^4)$, and this can be computed in $O(n^4 \log n)$ time.*

Lemma 5 *If h_i, h_{i+1} ($h_{i+1} > h_i$) are two consecutive critical heights in $L_{critical}$, then the topology of $H_{u(h)}(f)$ will be same for any $h \in [h_i, h_{i+1})$, where $u \in V$ and $f \in F$.*

Lemma 6 *It is not possible that the optimum height h^* is larger than the maximum height in the list $L_{critical}$.*

Lemma 7 *A pair of consecutive heights (h', h'') in $L_{critical}$ such that h' is not feasible but h'' is feasible, can be computed in $O(n^{k+3} k^2 \alpha^2(n) \log^2 n)$ time.*

Proof. We can find (h', h'') by performing a binary search on the sorted list $L_{critical}$. As each decision of the binary search takes $O(n^{k+3} k^2 \alpha^2(n) \log n)$ time (Lemma 3), the lemma follows. □

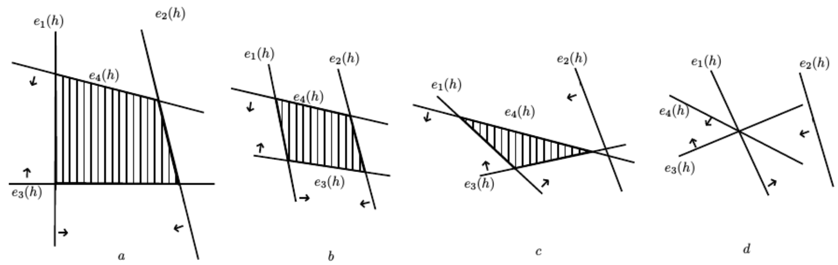

Fig. 3. Shrinking(a—d) of common invisibility region $\cap_{i=1}^{k} H_{u_i(h')}(f)$ due to increase in heights (arrow indicates the half-plane that is not visible)

4.2 Phase 2: Contact-height generation

Let $(u_1, u_2, \ldots, u_k) \in V^k$ be a guard k-tuple at the optimal height h^*. From above discussion, we know that $h^* \in (h', h'']$, and for a fixed watchtower base $u_i \in V$ and facet $f \in F$, the topology of invisibility region $H_{u_i(h)}(f)$ remain same for any height $h \in [h', h'')$ (see Lemma 5). Since $h^* > h'$, from Lemma 1, we know that at least one of the following happens:

- there exists a facet $f \in \mathcal{T}$ such that the common invisibility region $\cap_i^k H_{u_i(h')}(f)$ in not empty,
- there exists an edge $e \in E$ that has non-empty common invisibility interval, in other words, $\cap_{i=1}^{k} H_{u_i(h')}(e) \neq \emptyset$.

Note that the common invisibility region $\cap_{i=1}^{k} H_{u_i(h')}(f)$ is a collection of polygons. If we increase the height continuously from h' then the common invisibility region shrinks continuously, and at some specific height it converges into a point (see Fig. 3). When this event occurs, then at least three invisibility segments from $\cup_{i=1}^{k} H_{u_i(h')}(f)$ meet at a point.

Similarly, the common invisibility interval $\cap_{i=1}^{k} H_{u_i(h')}(e)$ is a collection of intervals along the edge e. If we increase the height continuously from h' then the common invisibility interval shrinks continuously, and at some specific height it converges into a point. When this event occurs, then at least two invisibility segments from $\cup_{i=1}^{k} H_{u_i(h')}(f_1) \bigcup \cup_{i=1}^{k} H_{u_i(h')}(f_2)$ meet at a point on the edge $e \in E$, where the facets f_1 and f_2 share the edge e.

In this phase, we enumerate each height $h \in (h', h'')$ where any of the following events occur:

Type-D Event: Any three invisibility segments e_1, e_2, e_3 from $\cup_{u_i \in V} H_{u_i(h')}(f)$ intersect at a point; e_1, e_2, e_3 must be due to at least two (at most three) watchtowers based at two (three) different vertices. Note that, if e_1, e_2, e_3 are due to a single watchtower then e_1, e_2, e_3 are three edges of an invisibility region $H_{u_i(h')}(f)$. This case is already considered as a Type-B event in Phase 1.

Type-E Event: Any two invisibility segments e_1, e_2 from $\cup_{u_i \in V} H_{u_i(h')}(f_1) \bigcup \cup_{u_i \in V} H_{u_i(h')}(f_2)$ and the edge e_3 intersect at a point, where e_3 is shared by both the facets f_1 and f_2; e_1, e_2 must be due to at least two watchtowers based at two different vertices. Note that if e_1, e_2 are due to a single watchtower then this case is already considered as a Type-C event in Phase 1.

We refer each height corresponding to the above two types of events as *contact height*. From the above discussions, it is easy to prove the following.

Lemma 8 *If $h^* \neq h''$, then h^* must be a contact height.*

Proof. Let $(u_1, u_2, \ldots, u_k) \in V^k$ be a guard k-tuple at the optimal height h^*. If $h^* \neq h''$, then the optimum height h^* depends on the following two cases; one of them is sure to occur.

Case 1 for a facet f, the common invisibility region $\cap_{i=1}^{k} H_{u_i(h')}(f)$ converges to a point, and for each facet $f' \in F \setminus f$ and edge $e' \in E$, we have $\cap_{i=1}^{k} H_{u_i(h')}(f') = \emptyset$ and $\cap_{i=1}^{k} H_{u_i(h')}(e') = \emptyset$.

Case 2 for an edge e, the common invisibility interval $\cap_{i=1}^{k} H_{u_i(h')}(e)$ converges to a point, and for each facet $f' \in F$ and edge $e' \in E \backslash e$, we have $\cap_{i=1}^{k} H_{u_i(h')}(f') = \emptyset$ and $\cap_{i=1}^{k} H_{u_i(h')}(e') = \emptyset$.

In Case 1, we know that at least three invisibility segments from $\cup_{i=1}^{k} H_{u_i(h')}(f)$ meet at a point. If more than three invisibility segments of $\cup_{i=1}^{k} H_{u_i(h')}(f)$ meet at a point, then each 3-tuple of those segments produce the same contact height h^*. With a similar argument, for Case 2, we can show that h^* is a contact height. □

Let e be an invisibility segment of $H_{u_i(h')}(f)$ due to the watchtower $u_i(h')$. Note that e must be along the line of intersection between the facet f and a plane defined by $u(h')$ and an edge $e' \in \mathcal{E}$. Recall that $\mathcal{E} = E \cup \{\text{line segment } \overline{v, v'} | v \in V\}$, where v' is a projection of v on the XY plane (assuming that all the points of V are above the XY plane). We use $e(h)$ to denote the position of the invisibility segment e at height $h \in (h', h'']$, i.e, $e(h)$ lies along the intersection of the facet f and the plane defined by the edge e' and the point $u_i(h)$. Using elementary geometry, we prove the following three lemmata.

Lemma 9 *Let $(a, b, c+h)$ be the Cartesian co-ordinates of the watchtower $u_i(h)$. Let $p_j = (a_j, b_j, c_j), j \in [2]$ be the two end points of the edge $e' \in \mathcal{E}$. Let $l\boldsymbol{x} + m\boldsymbol{y} + n\boldsymbol{z} = D$ be the equation of the plane containing the facet f. The vector form of equation of the line containing the segment $e(h)$ is:* $\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle = \langle 0, \frac{f_3(h) - \frac{C}{n} D}{f_2(h) - \frac{C}{n} m}, \frac{f_3(h) - \frac{f_2(h)}{m} D}{C - \frac{f_2(h)}{m} n} \rangle + t \langle n f_2(h) - mC, lC - n f_1(h), m f_1(h) - l f_2(h) \rangle$, *where*

$f_1(h) = (b - b_1)(c_2 - c_1) - (c + h - c_1)(b_2 - b_1)$,
$f_2(h) = (c + h - c_1)(a_2 - a_1) - (a - a_1)(c_2 - c_1)$,
$C = (a - a_1)(b_2 - b_1) - (b - b_1)(a_2 - a_1)$ *and* $f_3(h) = a_1 f_1(h) + b_1 f_2(h) + c_1 C$.

Proof. Let $\boldsymbol{P}$ be the plane defined by the point $u_i(h)$ and the edge e'. We have two vectors $\overrightarrow{p_1, u_i(h)} = \langle a - a_1, b - b_1, c + h - c_1 \rangle$ and
$\overrightarrow{p_1, p_2} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$ that lie completely in the plane $\boldsymbol{P}$.
So, the vector $\overrightarrow{n_1} = \overrightarrow{p_1, u_i(h)} \times \overrightarrow{p_1, p_2}$ is orthogonal to the plane $\boldsymbol{P}$. We have $\overrightarrow{n_1} = \langle f_1(h), f_2(h), C \rangle$, where
$f_1(h) = (b - b_1)(c_2 - c_1) - (c + h - c_1)(b_2 - b_1)$,
$f_2(h) = (c + h - c_1)(a_2 - a_1) - (a - a_1)(c_2 - c_1)$, and
$C = (a - a_1)(b_2 - b_1) - (b - b_1)(a_2 - a_1)$.
The equation of the plane $\boldsymbol{P}$ is $f_1(h)(\boldsymbol{x} - a_1) + f_2(h)(\boldsymbol{y} - b_1) + C(\boldsymbol{z} - c_1) = 0$, i.e.,

$$f_1(h)\boldsymbol{x} + f_2(h)\boldsymbol{y} + C\boldsymbol{z} = f_3(h) \tag{1}$$

Here $f_3(h) = a_1 f_1(h) + b_1 f_2(h) + c_1 C$ is a linear function of h.
We know that the equation of the plane $\boldsymbol{F}$ containing the facet f is

$$l\boldsymbol{x} + m\boldsymbol{y} + n\boldsymbol{z} = D \tag{2}$$

This planes have normal vector $\overrightarrow{n_2} = \langle l, m, n \rangle$. Let L denote the line of intersection of the two planes $\boldsymbol{P}$ and $\boldsymbol{F}$.
Then, the direction vector of the line L is: $\overrightarrow{v} = \overrightarrow{n_1} \times \overrightarrow{n_2} = \langle f_4(h), f_5(h), f_6(h) \rangle$, where $f_4(h) = nf_2(h) - mC$,
$f_5(h) = lC - nf_1(h)$ and
$f_6(h) = mf_1(h) - lf_2(h)$.
Now, we need to find a point p on the line L to construct the equation of the line. We consider p_0 to be a point on the line L with $x = 0$. Thus, substituting $x = 0$ in the equations 1 and 2 of the planes, we get $p_0 = (0, \dfrac{f_3(h) - \frac{C}{n}D}{f_2(h) - \frac{C}{n}m}, \dfrac{f_3(h) - \frac{f_2(h)}{m}D}{C - \frac{f_2(h)}{m}n})$.
Thus the vector form of the line L is $\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle = \overrightarrow{p_0} + t\overrightarrow{v}$
$= \langle 0, \dfrac{f_3(h) - \frac{C}{n}D}{f_2(h) - \frac{C}{n}m}, \dfrac{f_3(h) - \frac{f_2(h)}{m}D}{C - \frac{f_2(h)}{m}n} \rangle + t\langle f_4(h), f_5(h), f_6(h) \rangle$
$= \langle 0, \dfrac{f_3(h) - \frac{C}{n}D}{f_2(h) - \frac{C}{n}m}, \dfrac{f_3(h) - \frac{f_2(h)}{m}D}{C - \frac{f_2(h)}{m}n} \rangle + t\langle nf_2(h) - mC, lC - nf_1(h), mf_1(h) - lf_2(h) \rangle$. □

Lemma 10 *Let e_i be an invisibility segment of $H_{u_i(h')}(f)$ due to the watchtower $u_i(h')$, for each $i \in [3]$. The number of contact heights $h \in (h', h'')$ generated due to the meeting of the invisibility segments $e_1(h), e_2(h)$ and $e_3(h)$ at a single point can be at most three, and all of them can be enumerated in $O(1)$ time.*

Proof. According to Lemma 9, the vector form of equation of the line L_i on the plane F containing $e_i(h)$ is as follows:

$\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$
$= \langle 0, \dfrac{f_{i,3}(h) - \frac{C_i}{n}D}{f_{i,2}(h) - \frac{C_i}{n}m}, \dfrac{f_{i,3}(h) - \frac{f_{i,2}(h)}{m}D}{C_i - \frac{f_{i,2}(h)}{m}n} \rangle + t\langle nf_{i,2}(h) - mC_i, lC_i - nf_{i,1}(h), mf_{i,1}(h) - lf_{i,2}(h) \rangle$,

where each $f_{i,j}(h)$, $i, j \in [3]$ is a linear function of h.

Projecting L_i on the XY plane, we get the line L'_i as

$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \langle 0, \dfrac{f_{i,3}(h) - \frac{C_i}{n}D}{f_{i,2}(h) - \frac{C_i}{n}m} \rangle + t\langle nf_{i,2}(h) - mC_i, lC_i - nf_{i,1}(h) \rangle$.

Converting this vector form of equation in slope-intercept form, we get

$\boldsymbol{y} = \dfrac{lC_i - nf_{i,1}(h)}{nf_{i,2}(h) - mC_i}\boldsymbol{x} + \dfrac{f_{i,3}(h) - \frac{C_i}{n}D}{f_{i,2}(h) - \frac{C_i}{n}m}$
$= \dfrac{lC_i - nf_{i,1}(h)}{nf_{i,2}(h) - mC_i}\boldsymbol{x} + \dfrac{nf_{i,3}(h) - C_iD}{nf_{i,2}(h) - mC_i}$.

For the sake of simplicity, we rewrite this expression of L'_i as:

$\boldsymbol{y} = \dfrac{\phi_{i,1}(h)}{\phi_{i,3}(h)}\boldsymbol{x} + \dfrac{\phi_{i,2}(h)}{\phi_{i,3}(h)}$, where each $\phi_{i,j}(h)$, $i, j \in [3]$ is a linear function on h.

Now, to get the contact height h at which the invisibility segments $e_1(h), e_2(h)$ and $e_3(h)$ intersect at a single point, we find the equation of area of the triangle formed by L'_1, L'_2 and L'_3 and find roots of this equation.

We know that the area of triangle formed by three lines $\boldsymbol{y} = m_1\boldsymbol{x} + c_1$, $\boldsymbol{y} = m_2\boldsymbol{x} + c_2$ and $\boldsymbol{y} = m_3\boldsymbol{x} + c_3$ is given by

$$\Delta = \frac{\begin{vmatrix} -m_1 & 1 & -c_1 \\ -m_2 & 1 & -c_2 \\ -m_3 & 1 & -c_3 \end{vmatrix}^2}{2.K_1.K_2.K_3}$$

Where $K_1 = \begin{vmatrix} -m_2 & 1 \\ -m_3 & 1 \end{vmatrix}$, $K_2 = -\begin{vmatrix} -m_1 & 1 \\ -m_3 & 1 \end{vmatrix}$ and $K_3 = \begin{vmatrix} -m_1 & 1 \\ -m_2 & 1 \end{vmatrix}$ are the cofactors of $-c_1, -c_2, -c_3$, respectively, in the above matrix.

Using above formula, the area of triangle formed by L'_1, L'_2 and L'_3 is

$\Delta = \dfrac{\Phi_1(h)^2}{\Phi_2(h)\Phi_3(h)\Phi_4(h)}$, where $\Phi_1(h)$ is a cubic function in h, and $\Phi_2(h)$, $\Phi_3(h)$ and $\Phi_4(h)$ are quadratic function in h.

We need to solve one cubic equation $\Phi_1(h) = 0$ to find the heights where the area $\Delta = 0$. Thus, we have at most three distinct heights at which L_1, L_2 and L_3 intersect at a common point. □

Lemma 11 *Let e_1 be an invisibility segment of $H_{u_i(h')}(f_1)$ due to the watchtower $u_i(h')$, and e_2 be an invisibility segment of $H_{u_j(h')}(f_2)$ due to the watch-*

tower $u_j(h')$, $i \neq j$. Let $e_3 \in E$ be the edge that is shared by both the facets f_1 and f_2 . The number of contact heights $h \in (h', h'')$ generated due to the meeting of the invisibility segments $e_1(h), e_2(h)$ and the edge e_3 at a common point can be at most one, and it can be enumerated in $O(1)$ time.

Proof. Let $l_i\boldsymbol{x} + m_i\boldsymbol{y} + n_i\boldsymbol{z} = D_i$ be the equation of the plane F_i containing the facet f_i, $i \in [2]$. According to Lemma 9, the vector form of equation of the line L_i on the plane F_i containing $e_i(h)$ is as follows:
$\langle \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \rangle$
$= \langle 0, \dfrac{f_{i,3}(h) - \frac{C_i}{n_i} D}{f_{i,2}(h) - \frac{C_i}{n_i} m_i}, \dfrac{f_{i,3}(h) - \frac{f_{i,2}(h)}{m_i} D_i}{C_i - \frac{f_{i,2}(h)}{m_i} n_i} \rangle + t\langle n_i f_{i,2}(h) - m_i C_i, l_i C_i - n f_{i,1}(h), m_i f_{i,1}(h) - l_i f_{i,2}(h) \rangle$,
where each $f_{i,j}(h)$, $i \in [2], j \in [3]$ is a linear function of h.
Let the vector form of equation of the line containing the edge e_3 be $\langle x, y, z \rangle = \langle \alpha, \beta, \lambda \rangle + t\langle p, q, r \rangle$.
It is easy to see that $\rho_i = (\dfrac{\alpha f_i(h)}{f_i(h) - p}, \dfrac{\alpha q + \beta(f_i(h) - p)}{f_i(h) - p}, \dfrac{\alpha r + \lambda(f_i(h) - p)}{f_i(h) - p})$ is the intersection point of the two lines L_i and e_3, where $f_i(h) = n_i f_{i,2}(h) - m_i C_i$.
The length of the line segment $\overline{\rho_1, \rho_2}$ is $\tau(h) = \dfrac{\alpha\sqrt{q^2 + r^2 + 1}(f_2(h) - f_1(h))}{(f_1(h) - p)(f_2(h) - p)}$.
When L_1, L_2 and e_3 intersect at a common point then the length of $\tau(h) = 0$. So, we need to solve a linear equation: $f_2(h) - f_1(h) = 0$ on h to find the critical heights corresponding to this event. Thus the lemma follows. □

Lemma 12 *Total number of contact height is $O(n^7\alpha^3(n))$, and the sorted list $L_{contact}$ of all the contact heights can be enumerated in $O(n^7\alpha^3(n) \log n)$ time.*

Proof. For each triple of invisibility segments from $\cup_{u_i \in V} H_{u_i(h')}(f)$, we have at most three distinct contact heights due to Type-D events, and we can enumerate them in $O(1)$ time(From Lemma 10). Since the total number of invisibility segments in $\cup_{u_i \in V} H_{u_i(h')}(f)$ is $O(n^2\alpha(n))$, the total number of contact heights (due to Type-D events) for the facet f is $O(n^6\alpha^3(n))$. Considering all facets, total number of contact heights due to Type-D events is $O(n^7\alpha^3(n))$, and we can enumerate all of them in $O(n^7\alpha^3(n))$ time. Using similar argument and Lemma 11, it is easy to see that the total number of contact heights due to Type-E events is $O(n^5\alpha^2(n))$, and we can enumerate them in $O(n^5\alpha^2(n))$ time. As we need sorting to obtain the sorted list $L_{contact}$ containing all the contact heights, the total time complexity is $O(n^7\alpha^3(n) \log n)$. □

Theorem 13. *Given polyhedral terrain $\mathcal{T}$ in $\mathbb{R}^3$ with n vertices and a fixed integer k, we can find a guard k-tuple for the terrain $\mathcal{T}$ with minimum height in $O(n^{k+3}k^2\alpha^2(n) \log^2 n + n^7\alpha^3(n) \log n)$ time.*

5 Conclusion

We propose a simple to implement algorithm for k-watchtower problem, for any fixed integer k. Note that the time complexity for 3-watchtower problem using our algorithm is $O(n^7\alpha^3(n)\log n)$ where the dominating term is the number of contact heights enumerated in the phase 2 of the algorithm. A natural direction of future work is to improve the time complexity for $k = 3$. On the other hand, for $k > 3$, the time needed by the decision procedure is the bottleneck of our algorithm. Whether one can improve the time complexity of the decision procedure further remains open.

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