A Lower Bound on the Growth Constant of Polyaboloes on the Tetrakis Lattice*

Gill Barequet[1] and Minati De[2]

[1] Dept. of Computer Science
Technion—Israel Inst. of Technology
Haifa 3200003, Israel
barequet@cs.technion.ac.il

[2] Dept. of Mathematics
Indian Inst. of Technology Delhi
New Delhi, India
minati@maths.iitd.ac.in

Abstract. A "lattice animal" is an edge-connected set of cells on a lattice. In this paper we consider the *Tetrakis* lattice, and provide the first lower bound on λ_τ, the growth constant of polyaboloes (animals on this lattice), proving that $\lambda_\tau \geq 2.4345$. The proof of the bound is based on a concatenation argument and on calculus manipulations. If we also rely on an unproven assumption, which is, however, supported by empirical data, we obtain the conditional slightly-better lower bound 2.4635.

Keywords: Lattice animals, polyaboloes, concatenation, growth constant.

1 Introduction

Lattice animals are edge-connected sets of cells on a lattice. (One can regard the lattice as a graph. In the dual graph, that is, in the cell-adjacency graph of the original lattice, cells (faces) become sites (vertices). Hence, "site animals" are also a common name for lattice animals made of connected cells.)

The study of lattice animals began in the mid 1950s in the community of statistical physics. For example, Temperley [15] investigated the mechanics of macro-molecules, and Broadbent and Hammersley [6] studied percolation processes. Mathematicians began to show interest in lattice problems at about the same time. Harary [10] composed a list of unsolved problems in the enumeration of graphs, and Eden [8] analyzed cell growth problems.

Fixed animals are considered distinct if they have different *shapes* or *orientations*. In this paper we consider only fixed animals, hence, we omit hereafter the adjective "fixed" when we refer to lattice animals.

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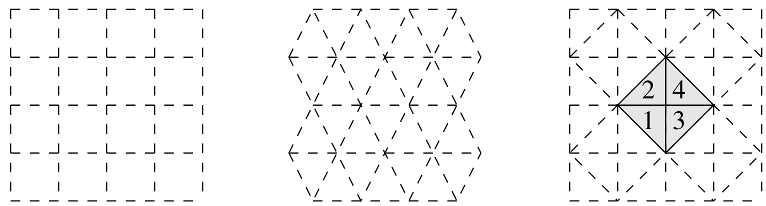


(a) Square lattice (b) Regular triangular lattice (c) Tetrakis lattice

Fig. 1. Two-dimensional lattices

Most attention was given in the literature to the cubical lattices in two dimensions (see Figure 1(a)) and higher dimensions. Less attention was given to other lattices, such as the triangular and hexagonal lattices in two dimensions.

The size of an animal is the number of cells it contains. Let (a_n) be an integer sequence. The limit $\lim_{n\to\infty} a_{n+1}/a_n$, if it exists, is called the *growth constant* of (a_n). In the context of sequences that enumerate lattice animals by size, the existence of a growth constant was discussed for the first time for the square lattice, on which animals are called *polyominoes* and the number of polyominoes of size n is denoted by $A(n)$. Klarner [11] showed that $\lambda := \lim_{n\to\infty} \sqrt[n]{A(n)}$ exists. Only three decades later, Madras [13] proved that $\lim_{n\to\infty} A(n+1)/A(n)$ exists, and, hence, is equal to λ. A main open problem in this area is to determine the exact value of this elusive constant. The currently best-known lower and upper bounds on λ are 4.0025 [5] and 4.6496 [12], respectively.

Similarly, a *polyiamond* of size n is an edge-connected set of n cells on the regular two-dimensional triangular lattice (Figure 1(b)), in which every cell is an equilateral triangle. Let $T(n)$ denote the number of polyiamonds of size n. The existential results of Klarner [11] and Madras [13] extend to all periodic lattice, and, in particular, to the triangular lattice. Hence, the sequence $T(n))$ has a growth constant, $\lambda_T := \lim_{n\to\infty} T(n+1)/T(n)$. The currently best-known lower and upper bounds on λ_T are 2.8424 [4] and 3.6108 [3], respectively.

In this paper, we consider animals on the Tetrakis lattice (see Figure 1(c)), also known as *kisquadrille* [7, §21]. Throughout the paper, we refer to these animals as "polyaboloes." It is important to note that we restrict ourselves to polyaboloes that can be embedded in the Tetrakis lattice. forbidding triangle neighborhoods like the one between the two dark-gray triangles in Figure 4(c). Let $\tau(n)$ denote the number of polyaboloes of size n. Figure 2 shows polyaboloes of size up to 3. The first twenty elements of $(\tau(n))$ appear as sequence A197467 in the On-Line Encyclopedia of Integer Sequences [1], referring to these animals as "poly-[4.8ˆ2]-tiles"; See also Grünbaum and Shephard [9, §§2.7,6.2,9.4]. More values of this sequence are provided in the Appendix of this paper. By Klarner [11] and Madras [13], we know that the sequence $(\tau(n))$ has a growth constant, that is, the limit $\lambda_\tau := \lim_{n\to\infty} \tau(n+1)/\tau(n) = \lim_{n\to\infty} \sqrt[n]{\tau(n)}$ exists. In this paper, we set a lower bound on λ_τ, showing that $\lambda_\tau \geq 2.4345$. Under some unproven assumption, which is, however, supported by empirical data, we obtain a conditional improved lower bound, $\lambda_\tau \geq 2.4635$.

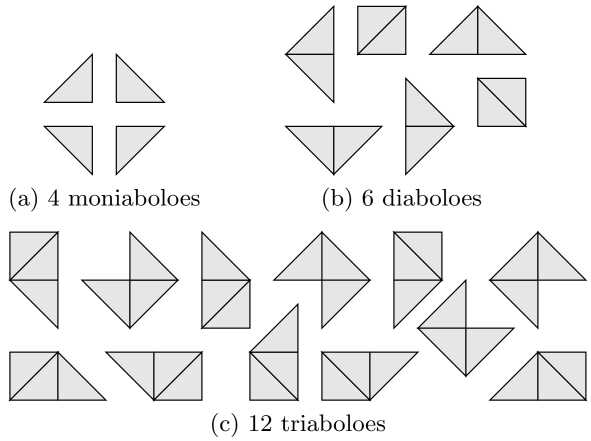


Fig. 2. Polyaboloes of size up to 3

2 Preliminaries

As is shown in Figure 1(c), the Tetrakis lattice contains cells of four distinct types, which we denote as cells of Type 1 through Type 4. We also define a lexicographic order on the cells of the lattice as follows: A triangle t_1 is *smaller* than triangle $t_2 \neq t_1$ if either

1. The column of t_1 is to the left of the column of t_2; or
2. Both t_1 and t_2 are in the same column, and t_1 lies below t_2.

We say that a polyabolo is of Type i (for $1 \leq i \leq 4$) if its lexicographically-*smallest* triangle is of Type i, and denote by $\tau_i(n)$ the number of such polyaboloes of size n. Furthermore, we split every Type-i polyaboloes into four complementing Subtypes (i, j) polyaboloes $(1 \leq j \leq 4)$, in which the lexicographically-smallest triangle is of Type i (as before), and the lexicographically-*largest* triangle is of Type j. We also denote by $\tau_{i,j}(n)$ (for $1 \leq i, j \leq 4$) the number of polyaboloes of Type (i, j) and size n.

A concatenation of two polyaboloes P_1, P_2 is the union of the cell set of P_1 and the cell set of a translated copy of P_2, such that the *largest* cell of P_1 is attached to the *smallest* cell of P_2, and all cells of P_1 are smaller than the translates of cells of P_2. It is easy to verify that the concatenation of a given pair of polyaboloes is either impossible or unique; see Figure 4 and Table 1.

3 Properties

First, we observe a close relationship between three of the types of polyaboloes.

Lemma 1. $\tau_3(n+2) \stackrel{(ii)}{=} \tau_4(n+1) \stackrel{(i)}{=} \tau_1(n)$ *(for $n \geq 1$).*

Proof. Both claims are verified by observing the Tetrakis lattice. If t, the smallest triangle of a polyabolo P, is of Type 4, then the only neighbor of t (within P) is the lattice triangle lying immediately above t, which is of Type 1. Hence, we can remove t from P, and obtain a unique Type-1 polyabolo of size less by one. This creates a bijection between n-cell polyaboloes of Type 4 and $(n-1)$-cell polyaboloes of Type 1. Claim (i) follows immediately.

Similarly, if t, the smallest triangle of a polyabolo P, is of Type 3, then the only neighbor of t (within P) is the lattice triangle lying immediately above t, which is of Type 4. Hence, we can remove t from P, and obtain a unique Type-4 polyabolo of size less by one. This creates a bijection between n-cell polyaboloes of Type 3 and $(n-1)$-cell polyaboloes of Type 4. Claim (ii) follows immediately as well. □

Second, we note that the sequences enumerating polyaboloes of each type are monotone increasing. (This is a universal property of animals on all lattices.)

Observation 1. *$\tau_i(n) \le \tau_i(n+1)$, for $1 \le i \le 4$ and all $n \in \mathbb{N}$.*

Indeed, consider n-cell polyaboloes of Type i. By stacking one more triangle on top of the lexicographically-largest triangle of each such polyabolo, we create polyaboloes of size $n+1$ of the same type and *without* repetitions. In fact, for each Type i of polyaboloes, there exists a nominal size $n_0 = n_0(i)$, such that for all $n \ge n_0$, there exist polyaboloes of size $n+1$ that cannot be built this way: All such polyaboloes whose largest triangle cannot be removed without breaking the polyabolo into two pieces. Hence, $\tau_i(n) < \tau_i(n+1)$, for $1 \le i \le 4$ and all $n \ge n_0(i)$.

Third, we find relations between the numbers of polyaboloes of different types.

Corollary 1. *$\tau_3(n) \le \tau_4(n) \le \tau_1(n)$ (for every $n \in \mathbb{N}$).*

Proof. The claim follows from Lemma 1 and Observation 1. □

Finally, we observe a simple equality of numbers of some polyaboloes of different subtypes.

Observation 2. *$\tau_{i,j}(n) = \tau_{5-j,5-i}(n)$, for $1 \le i, j \le 4$ and all $n \in \mathbb{N}$.*

Observation 2 is easily justified by rotating the plane by 180°.

4 The Bound

We are now ready to prove our main result, setting a lower bound on λ_τ.

Theorem 1. *$\lambda_\tau \ge 2.4345$.*

$j \setminus i$	1	2	3	4
1	0	1	1	0
2	0	0	1	1
3	0	0	0	1
4	1	0	0	0

Table 1. Number of valid concatenation options for all cases of triangle j with triangle i

Proof. We proceed with a concatenation argument tailored to the specific lattice under consideration. Note that the only valid concatenation options are those specified in Table 1. Hence, the only families of polyaboloes which are closed under concatenation are those of Types (2,1), (3,1), (3,2), (4,2), (4,3), and (1,4). Consider one such family of Type (i, j) (the exact values of i, j will be determined later). By the same arguments used by Klarner [11] and Madras [13], we know that the sequence $(\tau_{i,j}(n))$ has a growth constant. In addition, we obviously have that $\tau_{i,j}(n) \le \tau(n)$. Using elementary facts from calculus, we conclude that the growth constant of $(\tau_{i,j}(n))$, which we denote by $\lambda_{\tau_{i,j}}$, is at most λ_τ. Therefore, any lower bound we set on $\lambda_{\tau_{i,j}}$ will also be a lower bound on λ_τ.

Let P_1, P_2 be two polyaboloes of Type (i, j) and size n, and let Q be the polyabolo of size $2n$ that is the result of concatenating P_1 and P_2. It is crucial to observe that Q *cannot* be the result of concatenating any other pair of polyaboloes, both of size n (but it may be represented as the concatenation of polyaboloes of different sizes). However, there exist polyaboloes of size $2n$, which cannot be represented as the concatenation of any pair of polyaboloes of size n, because their lexicographically-smallest (or largest) n triangles do not form a connected set of triangles; see, for example, Figure 3. (In our setting, there is

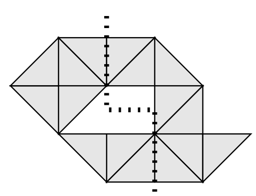


Fig. 3. A sample 16-triangle polyabolo which cannot be represented as the concatenation of two 8-triangle polyaboloes

another possible reason for this: It may be the case that the lexicographically-smallest (or largest) n triangles of a polyabolo of Type (i, j) and size $2n$ is not a polyabolo of the same type.) Hence,

$$\tau_{i,j}^2(n) \le \tau_{i,j}(2n).$$

By simple manipulations of this relation, we obtain that

$$\big(\tau_{i,j}(n)\big)^{1/n} \le \big(\tau_{i,j}(2n)\big)^{1/(2n)}.$$

Thus, $\big(\tau_{i,j}(k)\big)^{1/k}, \big(\tau_{i,j}(2k)\big)^{1/(2k)}, \big(\tau_{i,j}(4k)\big)^{1/(4k)}, \ldots$ is monotone increasing for any value of k, and, as a subsequence of $\Big(\big(\tau_{i,j}(n)\big)^{1/n}\Big)$, it converges to $\lambda_{\tau_{i,j}}$ as well. Therefore, any term of the form $\big(\tau_{i,j}(n)\big)^{1/n}$ is a lower bound on $\lambda_{\tau_{i,j}}$. (See the Appendix for some of the available values of the sequences $(\tau_{i,j}(n))$.) Now, we simply choose the values of (i,j) that provide the best lower bound out of the six options. It turns out that $(1,4)$ is the best choice. In particular, $\lambda_{\tau_{1,4}} \ge (\tau_{1,4}(36))^{1/36} = 81530581108477^{1/36} \ge 2.4345$, and the claim follows. □

5 A Conditional Better Bound

If we refine the proof in the previous section to consider simultaneously all possible concatenation options, we may achieve a better (higher) lower bound on λ_τ. However, the refined proof requires a reasonable assumption which we were unable to prove so far.

Assumption 1. *$\tau_1(n) \ge \tau_2(n)$ for all $n \in \mathbb{N}$.*

Theorem 2. *Under Assumption 1, we have that $\lambda_\tau \ge 2.4635$.*

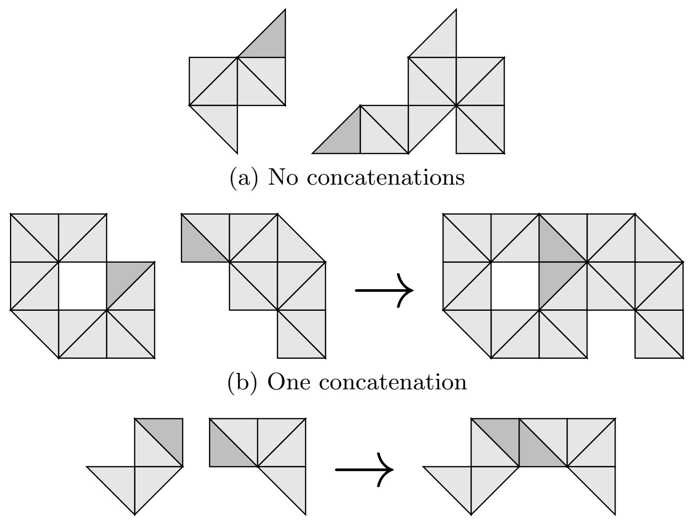
Proof. As in the proof of Theorem 1, we proceed with a concatenation argument. Since not all pairs of polyaboloes of size n can be concatenated (as can be seen in Table 1), let us count systematically those pairs of polyaboloes that can be concatenated. It can easily be verified that pairs of polyaboloes can be concatenated in at most one way. Here are two examples.

– A polyabolo whose largest triangle is of Type 2 cannot be concatenated at all with a polyabolo whose smallest triangle is of Type 1 (see Figure 4(a)).
– A polyabolo whose largest triangle is of Type 3 can be concatenated in one way with a polyabolo whose smallest triangle is of Type 4 (see Figure 4(b)).

Note that some "plausible" concatenations are in fact not allowed on our lattice. For example, there is no way to concatenate a polyabolo whose largest triangle is of Type 1 with a polyabolo whose smallest triangle is of Type 4. Indeed, the former triangle can be attached to the latter triangle either to the left of it or below it, but neither composition is valid (see Figure 4(c)).

By rotational symmetry, the number of polyaboloes of size n, whose *largest* (top-right) triangle is of Type j, is $\tau_{5-j}(n)$. Indeed, such a rotation converts triangles of Type 1 to triangles of Type 4 (and vice versa), and triangles of Type 2 to triangles of Type 3 (and vice versa). Therefore, the total number of concatenations of two polyaboloes of size n is

$$\tau_4(n)\tau_2(n) + \tau_4(n)\tau_3(n) + \tau_3^2(n) + \tau_3(n)\tau_4(n) + \tau_2(n)\tau_4(n) + \tau_1^2(n)$$
$$= \tau_1^2(n) + \tau_3^2(n) + 2\tau_4(n)(\tau_2(n) + \tau_3(n)).$$



(a) No concatenations

(b) One concatenation

(c) An invalid concatenation

Fig. 4. Valid and invalid concatenations

Let P_1, P_2 be two polyaboloes of size n, which can be concatenated in some way ξ (one of the options listed in Table 1), and yield a polyabolo Q of size $2n$. As observed in the proof of Theorem 1, the polyabolo Q can be represented as the concatenation of two polyaboloes of size n only by the triple (P_1, P_2, ξ). However, there exist polyaboloes of size $2n$, which cannot be represented as the concatenation of two polyaboloes of size n, because their lexicographically-smallest (or largest) n triangles do not form a connected set of triangles. Hence, we conclude that

$$\tau_1^2(n) + \tau_3^2(n) + 2\tau_4(n)(\tau_2(n) + \tau_3(n)) \leq \tau(2n). \tag{1}$$

Let us now find a good lower bound on the number of concatenations. Denote by $x_i(n)$ (for $1 \leq i \leq 4$) the fraction of polyaboloes of Type i out of all polyaboloes of size n, that is, $x_i(n) = \tau_i(n)/\tau(n)$. The left-hand side of Equation (1) can be rewritten as

$$\big(x_1^2(n) + x_3^2(n) + 2x_4(n)(x_2(n) + x_3(n))\big)\tau^2(n). \tag{2}$$

Obviously, we have that

$$x_4(n) = 1 - x_1(n) - x_2(n) - x_3(n). \tag{3}$$

Substituting Equation (3) into the count of concatenations (Equation (2)), we find that the left-hand side of Relation (1) is equal to

$$\big(x_1^2(n) + x_3^2(n) + 2(1 - x_1(n) - x_2(n) - x_3(n))(x_2(n) + x_3(n))\big)\tau^2(n).$$

Our next goal is to find a lower bound on the trivariate function

$$f(x_1, x_2, x_3) := x_1^2 + x_3^2 + 2(1 - x_1 - x_2 - x_3)(x_2 + x_3)$$

in the range $[0, 1] \times [0, 1] \times [0, 1]$. (In fact, the range is open on all sides since there are polyaboloes of all four types for all values of n.) This minimization problem is subject to the following constraints:

1. $x_1 + x_2 + x_3 \leq 1$: Obviously, the number of polyaboloes of Types 1, 2, and 3 cannot exceed the number of all polyaboloes. (Since all x_is are nonnegative, this constraint is actually weaker than constraints 2 and 3.)
2. $x_3 \leq x_4$, that is, $2x_3 \leq 1 - x_1 - x_2$: This follows from Corollary 1.
3. $x_4 \leq x_1$, that is, $1 - x_2 - x_3 \leq 2x_1$: This also follows from Corollary 1.

Subject to the above three constraints, the function $f(x_1, x_2, x_3)$ assumes at $(0, 1, 0)$ a minimum of 0, which is useless. Thus, we need to add a constraint which keeps x_2 away from 1, or keeps either x_1 or x_3 away from 0. Empirically (see the Appendix), we see that the sequences $(x_1(n))$ and $(x_2(n))$ are monotone increasing (and the limits of both are visually around 0.4), while $x_4(n)$ and $x_3(n)$ are monotone decreasing (and their limits are visually around 0.15 and 0.05, respectively), and the existing data suggest that $x_1(n) > x_2(n) > x_4(n) > x_3(n)$ for $n \geq 3$. If we rely on Assumption 1 and add to the above the additional constraint $x_1 \geq x_2$, the function $f(x_1, x_2, x_3)$ now assumes a minimum of 1/4 at two points: (0.5,0.5,0) and (0.5,0,0). (The second minimum point seems to be superfluous.) Hence,

$$\frac{1}{4}\tau^2(n) \leq \tau(2n).$$

By simple manipulations of this relation, we obtain that

$$\left(\frac{1}{4}\tau(n)\right)^{1/n} \leq \left(\frac{1}{4}\tau(2n)\right)^{1/(2n)}.$$

This implies that the sequence $\left(\frac{1}{4}\tau(k)\right)^{1/k}, \left(\frac{1}{4}\tau(2k)\right)^{1/(2k)}, \left(\frac{1}{4}\tau(4k)\right)^{1/(4k)}, \ldots$ is monotone increasing for any value of k, and, as a subsequence of $\left(\left(\frac{1}{4}\tau(n)\right)^{1/n}\right)$, it converges to λ_τ as well. Therefore, any term of the form $\left(\frac{1}{4}\tau(n)\right)^{1/n}$ is a lower bound on λ_τ. (See the Appendix for available values of the sequence $(\tau(n))$.) In particular, $\lambda_\tau \geq (\frac{1}{4}\tau(36))^{1/36} = (499003797597583/4)^{1/36} \geq 2.4635$, and the claim follows. $\square$

6 Conclusion

In this paper, we prove a lower bound on the growth constant of polyaboloes on the Tetrakis lattice, namely, that $\lambda_\tau \geq 2.4345$. Under some empirical assumption, we obtain an improved conditional lower bound of 2.4635. Future work includes improving the lower bound and finding a good upper bound on λ_τ.

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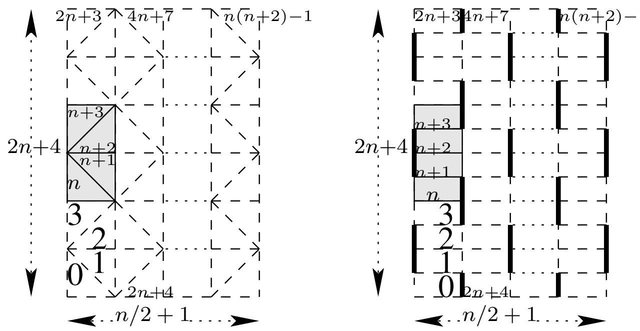
Appendix: Computing Elements of $(\tau(n))$

We have implemented Redelmeier's algorithm [14] for counting polyominoes, and adapted it to the Tetrakis lattice.[3] The algorithm was implemented in C on a server with four 2.20GHz Intel Xeon processors and 512GB of RAM. The software consisted of about 200 lines of code.

Assume, for simplicity, that we wanted to count polyaboloes up to size n, where n is divisible by 4. Then, we created the graph, dual of the portion of the Tetrakis lattice, that consists of $n/2$ columns, each of height $2n+4$. Cells

[3] Originally, the algorithm was proposed for counting polyominoes (site animals on the square lattice). However, as was already noted elsewhere (see, e.g., [2]), this algorithm can be adapted to any lattice, once it is formulated as an algorithm for counting connected subgraphs of a given graph, that contain one marked vertex in the graph. The reader is referred to the reference cited above for further details.



(a) (b)

Fig. 5. Cell IDs for Redelmeier's Algorithm

(triangles) of this portion of our lattice were numbered as is shown in Figure 5(a). In fact, the cell-adjacency graph was identical to the one shown in Figure 5(b), where a thick edge means that the two cells sharing this edge were not considered as neighbors. In order to count polyaboloes of Types 1, 2, 3, or 4, we fixed their smallest triangle at cell number $n+1$, $n+2$, $n+3$, or n, respectively. This ensured that animals of size n would never spill over the allocated portion of the Tetrakis lattice.

In fact, we needed to count only polyaboloes of two out of the four types, as is implied by Lemma 1. Thus, we ran our program for counting polyimaonds of Types 1 and 2, and computed counts of polyimaonds of Types 3 and 4 by applying the theorem. Then, we summed up the results to finally obtain $\tau(n) = \sum_{i=1}^{4} \tau_i(n)$. The running times of our program were 27.5 and 26.25 days, for computing $\tau_1(n)$ and $\tau_2(n)$ for $1 \le n \le 36$, respectively, for a total of 53.75 days for computing $\tau(n)$ for this range of n.

Table 2 provides the split of $\tau(36)$ into all 16 subtypes. Table 3 shows the total counts of polyaboloes, produced by our program, extending significantly the previously-published counts [1, Sequence A197467]. Figure 6 plots the known values of $\sqrt[n]{\tau(n)}$ and $\tau(n)/\tau(n-1)$ for $2 \le n \le 36$, demonstrating the convergence of the two sequences. Figure 7 plots the 36 known values of the sequences $(x_i(n))$ $(i = 1, \ldots, 4)$, showing the tendencies of these sequences.

$i \setminus j$	1	2	3	4	Total
1	30,137,895,510,778	11,148,998,271,068	78,876,705,335,638	81,530,581,108,477	201,694,180,225,961
2	29,162,541,569,420	10,790,377,951,489	76,306,266,133,179	78,876,705,335,638	195,135,890,989,726
3	4,127,703,521,100	1,529,507,765,873	10,790,377,951,489	11,148,998,271,068	27,596,587,509,530
4	11,148,998,271,068	4,127,703,521,100	29,162,541,569,420	30,137,895,510,778	74,577,138,872,366
Grand Total					**499,003,797,597,583**

Table 2. Split of $\tau(36)$ into the 16 subtypes $\tau_{i,j}(36)$, $1 \le i, j \le 4$

n	$\tau_1(n)$	$\tau_2(n)$	$\tau_3(n)$	$\tau_4(n)$	$\tau(n)$
1	1	1	1	1	4
2	2	2	1	1	6
3	5	4	1	2	12
4	10	8	2	5	25
5	22	19	5	10	56
6	52	48	10	22	132
7	125	121	22	52	320
8	311	304	52	125	792
9	781	759	125	311	1,976
10	1,965	1,905	311	781	4,962
11	4,986	4,844	781	1,965	12,576
12	12,765	12,424	1,965	4,986	32,140
13	32,904	32,049	4,986	12,765	82,704
14	85,303	83,072	12,765	32,904	214,044
15	222,145	216,224	32,904	85,303	556,576
16	580,700	565,062	85,303	222,145	1,453,210
17	1,523,496	1,482,251	222,145	580,700	3,808,592
18	4,010,346	3,900,592	580,700	1,523,496	10,015,134
19	10,587,019	10,292,607	1,523,496	4,010,346	26,413,468
20	28,019,133	27,227,765	4,010,346	10,587,019	69,844,263
21	74,323,315	72,197,057	10,587,019	28,019,133	**185,126,524**
22	197,565,811	191,849,795	28,019,133	74,323,315	**491,758,054**
23	526,189,451	510,796,099	74,323,315	197,565,811	**1,308,874,676**
24	1,403,920,681	1,362,392,571	197,565,811	526,189,451	**3,490,068,514**
25	3,751,867,755	3,639,699,653	526,189,451	1,403,920,681	**9,321,677,540**
26	10,041,587,514	9,738,372,232	1,403,920,681	3,751,867,755	**24,935,748,182**
27	26,912,890,591	26,092,611,572	3,751,867,755	10,041,587,514	**66,798,957,432**
28	72,223,625,842	70,002,807,553	10,041,587,514	26,912,890,591	**179,180,911,500**
29	194,053,148,466	188,035,944,757	26,912,890,591	72,223,625,842	**481,225,609,656**
30	521,974,915,118	505,660,330,038	72,223,625,842	194,053,148,466	**1,293,912,019,464**
31	1,405,512,260,944	1,361,250,747,068	194,053,148,466	521,974,915,118	**3,482,791,071,596**
32	3,788,326,126,815	3,668,175,811,997	521,974,915,118	1,405,512,260,944	**9,383,989,114,874**
33	10,220,263,525,941	9,893,931,070,016	1,405,512,260,944	3,788,326,126,815	**25,308,032,983,716**
34	27,596,587,509,530	26,709,792,413,846	3,788,326,126,815	10,220,263,525,941	**68,314,969,576,132**
35	74,577,138,872,366	72,166,102,653,759	10,220,263,525,941	27,596,587,509,530	**184,560,092,561,596**
36	201,694,180,225,961	195,135,890,989,726	27,596,587,509,530	74,577,138,872,366	**499,003,797,597,583**

Table 3. Counts of polyaboloes (values of $\tau(21)$–$\tau(36)$ (in bold) are new)

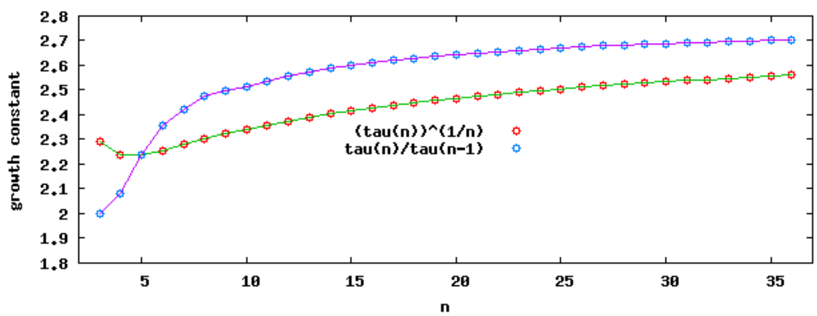


Fig. 6. Convergence of $\sqrt[n]{\tau(n)}$ and $\tau(n)/\tau(n-1)$

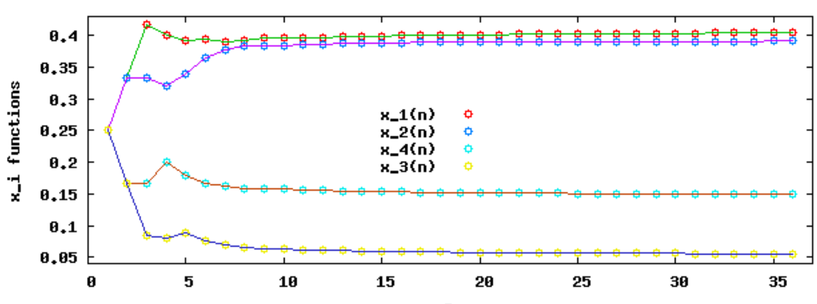


Fig. 7. Empirical monotonicity and convergence of $(x_i(n))$ (for $1 \leq i \leq 4$)