Math 2E03- Introduction to Modelling

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- Population ecology+genetics $\longrightarrow$ population Biology
- Population ecology is the branch of ecology that studies the struture and dynamics of populations.
- Pop. Biology is simply the study of biological populations. The goals of P. bio. are to understand and predict the dynamics of pop. Understanding, explaining and predicting dynamics of biological populations will require models , models that are expressed in the language of mathematics. Here we emphasize the role of models in understanding the pop. bio.

- One of the most succesful theories in pop. bio. has been that of the dynamics of age structured pop. growth. Given information about the age at which individuals have offspring and the probability of death at diffferent ages, we can make detailed predictions about the long term changes in population number.

Density independent Population Growth:

Here we examine the simplest models of pop. growth: those which assume density independence. We say that the growth of a pop. is density independent if the birth and death rates per individual do not depnd on the pop. size. We will look at both species with overlapping generations, like humans, and those with discrete generations, like many butterflies

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Continuous time We begin with species in which generation overlap and for which births can occur at any time, as in humans. Then it makes sense to predict or count the population size at all times, rather than at specified intervals. Thus we will use a continuous time model.

Mathematical aspects of continuous time model:

Let b be the birth rate per capita and m death rate per capita. The rate of changes of the total number of individuals, $\frac{dN}{dt}$, is given by the rate of births in population minus the rate of deaths. The rate of births in the poipulation is given by the per capita birth rate times the number of individuals, or bN. Similarly ,the population level death rate is given by mN. Thus

$$\frac{dN}{dt} = bN - mN \tag{1}$$

This equation is simply written as

$$\frac{dN}{dt} = rN \tag{2}$$

where $r = b - m$. Here r, the rate of increase, depends on both the birth rate and the death rate. Now we are using seperation of varables techniques for solving differential equation.

Mathematical aspects of continuous time model:

We first seperate the varables by putting all the terms with N on one side of the equation and all the terms t on the other side of the equation.

(3) $$\frac{dN}{N} = rdt$$

Integrate both side of the equation from $t = 0$ to $t = T$,

(4) $$\int_{t=0}^{t=T} \frac{dN}{N} = \int_{t=0}^{t=T} rdt$$

Compute the integrals to find

(5) $$\ln N(t)|_{t=0}^{t=T} = rt|_{t=0}^{t=T}$$

(6) $$\ln(N(T)) - ln(N(0)) = rT$$

Mathematical aspects of continuous time model:

Taking the expontial of boyh sides

(7) $$e^{\ln(N(T))}e^{-\ln(N(0))} = e^{rT}$$

Finally, solve for $N(T)$

(8) $$\frac{N(T)}{N(0)} = e^{rT}$$

(9) $$N(T) = N(0)e^{rT}$$

This model predicts that

- if $r = 0$, the population size is stationary
- if $r > 0$, the population grows expontially without bound.
- if $r < 0$, the population approaches 0

Mathematical aspects of continuous time model:

Do biological pop. ever demonstrate expontial growth?...................

Second, an equilibrium pop. attained when r is exactly 0...................

The qualitative behavior of the solution is determined only by the sign of r.

Mathematical aspects of discrete time model:

Butterflies breed once per year, laying their eggs close to April 1. Adults fly only for a short period, and then die. For these species, a model which assumes both that births occur continuously and that generate overlap is inappropriate.
For univoltine (one gentration per year) insects , a discrete time model, with population measurement only taken at fixed times, is more appropriate. We will measure time in units of generations, which may be one year. Here let R be the number of individuals in the next genration per individuals in the current generation. Thus, if N_t is the number of individuals in the population at generation t,

$$N_{t+1} = RN_t \tag{10}$$

From the relationship

$$N_2 = RN_1 = R(RN_0) = R^2N_0 \tag{11}$$

we conclude that

$$N_t = R^tN_0 \tag{12}$$

Relationship between continuos and discrete models:

We are concluding

$$R = e^r \text{ or } \ln(R) = r \tag{13}$$

if r is small, we can use Taylor series to find the approximate relationship

$$R \approx 1 + r \tag{14}$$

Expontial growth in nature In the laboratory, estimates of r or R can show quite rapid population growth, but in natural pop. r is almost near 0 and R near one. As we have noted, a population that does not show such values of growth will either explode in numbers or disappear rapidly.
One example of expontial growth in nature is illustrated by the following figure but after some years, the growth was no longer expontial.

The fundamental question of population biology is to determine the causes and consequences of the deviation from expontial growth, or simply what regulates populations.

Matrix and Vectors:

We will find matrices and vectors useful for expressing many of the models we study, as a compact notation for expressing linear equations.

- A matrix is a rectangular array of numbers, for example

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$$

- A row vector is a matrix with only 1 row.

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix}_{1\times 2}$$

- A coulmn vector is a matrix with only 1 column.

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}_{2\times 1}$$

Matrix addition and subtraction: The genral rule for adding or subtracting matrices of vectors is that corresponding elements are added or subtracted. Thus,

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}_{2\times 2} = \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} \\ a_{21}+b_{21} & a_{22}+b_{22} \end{pmatrix}_{2\times 2}$$

Multiplying a matrix by a scalar

$$\lambda \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2} = \begin{pmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{pmatrix}_{2\times 2}$$

Identity matrix The identity matrix with two rows and columns is

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}$$

Eigenvalues and Eigenvectors

if A is an $n \times n$ matrix, then a nonzero vector x in R^n is called and eigenvector of A if Ax is a scalar multiple of x, that is

$$Ax = \lambda x$$

for some scalar λ. The scalar λ is called an eigenvalue of A, and x is said to be and eigenvector of A corresponding to λ. Equivalently

$$(\lambda I - A)x = 0$$

For λ to be an eigenvalue, there must be nonzero solution of this equation. This equation has a nonzero solution iff

$$det(A - \lambda I) = 0$$

Therefore solution of equation $\|(A - \lambda I)\| = 0$ is called the eigenvalues of matrix A.