Math 2E03- Introduction to Modelling

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A Survey of models containing DEs

The rate at which the size, population temprature, etc. of objects in a system grow with time is often the important factor in modelling that system raising two elelphants in a field is quite different from raising two rabbits in a field since rabbits multiply far, ar more rapidly than do elephants and their large growth rate would cause them to soon overrun the area and to begin to starve and fight each other for food . In either case we would have set up the same differential equation to model their growth rate:

$$\frac{dp}{dt} = rp \tag{1}$$

where r is the growth rate.

Ordinary differential equation are those whose functions may change with respect to one variable, whereas partial differential equations are those whose function changes with respect to two or more variables.

Order: The order of a diferential equation is the order of the highest order derivatives present in the equation.

A linear differential equation is one in which the dependent variable y and its derivative appear in the additive combination of their first powers.

Bridging the DCE/DE Gap

Models containing DCEs are most common for banks to compound interest monthly, daily and yearly. Let's write the model of compounded interest (as described earlier)

$$a_n = a_{n-1} + \frac{i}{N} a_{n-1} \tag{2}$$

We wish to change over to continual function of t where t is measured in years. Let $A(t) = a_n$. Previously, it was shown that if interest is compounded N times in a year, then $n = Nt$, or $t = n/N = hn$, where $h = 1/N$ (h represent the times that passes between updates). Now, the term a_{n+1} reprsents the values one compounding period later 'h' times has passed. Thus $a_{n+1} = A(t+h)$. We may write the above equation as

$$A(t+h) - A(t) = ihA(t) \tag{3}$$

Let's devide both sides by h and see what happens as the compounding becomes an infinite number of times per year (Continuous)...in the limit sense $N \to \infty$, $h \to 0$. So our equation becomes

$$\begin{aligned} \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} &= \lim_{h \to 0} iA(t) \\ \frac{dA(t)}{dt} &= iA(t) \end{aligned} \tag{4}$$

Relation between geometric growth and exponential growth ?

Climbing the Learning curve

Jessica buys herself a violin and signs up for lessons. At first she knows absolutely nothing but she soon learns some scales and 'Twinkle Twinkle Little Star' . As the year grind she practices, gets better, the songs become more complex, and she begins to learn specific techniques. In short she practices more and more and the quality of the music that she produces improves.

She is now first violinist in an orchestra. She parctices to death day by day, concentarting on tiny details, but her day to day improvement is not increasing dramatically

Let's look at the situation : the better she gets the slower her improvement- more specifically, the less difference between her maximum possible performance M and her actual performance P, the slower her improvement

$$\frac{dP}{dt} = k(M - P) \tag{5}$$

Radioactive Decay Kinetics

Radioactive decay is random process. The amount of radiation or original isotope lost is propotional to the amount of isotope present. So the model will be

$$\frac{dX}{dt} = -\lambda X \tag{6}$$

and the solution is $X = X_0 e^{-\lambda t}$.

Isotope half time The time taken for a pariticular isotope to decay to half of its original amount. Thiis is constant for a particular isotope. So half time is

$$\begin{aligned} t_{\frac{1}{2}} &= \frac{\log(2)}{\lambda} \\ \lambda &= \frac{\log(2)}{t_{\frac{1}{2}}} \end{aligned} \tag{7}$$

Problem : A sample of wood is found to contain carbon with a specific activity of $7.8 \times 10^3 Bqkg^{-1}$ of carbon. Wood from apice of furniture made in 1897 has a specific activity of $14.7 \times 10^3 Bqkg^{-1}$ of acrbon when measured in 1997. Estimate the age of sample. The half life of carbon is 5730 years.

Solution hint : $\frac{X_1}{X_2} = e^{-\lambda(t_1 - t_2)}$. We get $t_1 = 5338.8$ years.

Production of Radioisotopes by Neutron Bombardment

Through bombardment with thermal neutrons, one may actually create radioisotopes from stable atoms. The rate of creation may be constant, but at the same time the radioisotopes begin to decay once they are formed. The results in the following ODE

$$\frac{dX}{dt} = k - \lambda X \tag{8}$$

And the solution is $X = \frac{k}{\lambda}(1 - e^{-\lambda t})$