Math 2E03- Introduction to Modelling

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Newton's Law of Cooling

Someone determined that the rate of change of the temprature of an object (like a glass of milk) is propotional to the difference in temprature of the object and its surrounding. If we let the temprature of the milk be T, time t, and the environmental temprature T_a, then we have

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$$\frac{dT}{dt} = -k(T - T_a) \tag{2}$$

Problem: Reena, a fussy hot chocolate drinker, wants her water to be precisely 85^oC, but she usually forgets and lets it boils. Having broken het thermometer, she asks you to calculate how long she should wait for it to cool from 100^oC to 85^oC. Assume that ambient temprature is constant. Can you solve her problem? if so, do so. If not, why not?

Solution:

Problem: Reena, a fussy hot chocolate drinker, wants her water to be precisely 85^oC, but she usually forgets and lets it boils. Having broken het thermometer, she asks you to calculate how long she should wait for it to cool from 100^oC to 85^oC. Assume that ambient temprature is constant. Can you solve her problem? if so, do so. If not, why not?

Solution:

Let's try for the solution of ODE for this model.

$$T = (T_0 - T_a)e^{-kt} + T_a \tag{3}$$

and the value of constant k can be find out by

$$k = \frac{1}{t}\log(\frac{T_0 - T_a}{T - T_a}) \tag{4}$$

Problem: Reena, a fussy hot chocolate drinker, wants her water to be precisely $85^o C$, but she usually forgets and lets it boils. Having broken het thermometer, she asks you to calculate how long she should wait for it to cool from $100^o C$ to $85^o C$. Assume that ambient temprature is constant. Can you solve her problem? if so, do so. If not, why not?

Solution:

Analysis of the result: This is important lesson -just because you can derive a theoritical ODE model and solve it, does not mean it will automatically be useful. The value of parameters and integration constants are determined by the specific system and conditions, and can in general only be determined empirically.

Problem: You have just returned home at 5pm on a hot day (35^oC) after a shopping and are expecting some guests for dinner. Unfortunately you forgot to put the pop into refrigerator before you left. The thermometer in the refrigrator displays the temprature in the refrigerator to be 5^oC. After 10 minutes in the refrigerator you observe that the pop has cooled to 28^oC. Will the pop have colled enough (say to 10^oC) in the remaining 80 minutes before your guests arrive? Or will you have to resort to putting the pop cans into the freezer (whose temprature is -2^oC) and run the risk of forgetting them yet again -recall those time pop froze and burst their cans! What time should you set your alarm for in order to remind you when to take the pop out of freezer, should you go that route.

Solution :

(5) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

Solution :

(7) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

$T = 28^oC$ at $t = 10$, $T_0 = 35^oC$, $T_a = 5^oC$

Solution :

(9) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

$T = 28^o C$ at $t = 10$, $T_0 = 35^o C$, $T_a = 5^o C$

$k \approx .0266$

Solution :

$$k = \frac{1}{t} \log \left(\frac{T_0 - T_a}{T - T_a} \right) \tag{11}$$

$T = 28^o C$ at $t = 10$, $T_0 = 35^o C$, $T_a = 5^o C$

$k \approx .0266$

We can now substitute in the time that the guest will arrive and see what the temprature of the pop will be

$$T = (T_0 - T_a)e^{-kt} + T_a \tag{12}$$

Solution :

$$k = \frac{1}{t} \log \left(\frac{T_0 - T_a}{T - T_a} \right) \tag{13}$$

$T = 28^o C$ at $t = 10$, $T_0 = 35^o C$, $T_a = 5^o C$
$k \approx .0266$
We can now substitute in the time that the guest will arrive and see what the temprature of the pop will be

$$T = (T_0 - T_a)e^{-kt} + T_a \tag{14}$$

$t = 90$, $T_0 = 35^o C$, $T_a = 5^o C$
$T \approx 7.74$

Solution :

(15) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

$T = 28^oC$ at $t = 10$, $T_0 = 35^oC$, $T_a = 5^oC$
$k \approx .0266$
We can now substitute in the time that the guest will arrive and see what the temprature of the pop will be

(16) $$T = (T_0 - T_a)e^{-kt} + T_a$$

$t = 90$, $T_0 = 35^oC$, $T_a = 5^oC$
$T \approx 7.74$ It is less than 10^oC, so we can leave in the fridge.

We can, then solve for the time t at which they will reach 10^oC and set our alarm clock for this time to remind us to take out. Now the question is $t = ?$

We can, then solve for the time t at which they will reach 10^oC and set our alarm clock for this time to remind us to take out. Now the question is $t =?$

$T_0 = 28^oC$, $T_a = -2^oC$, $T = 10^oC$ so $t \approx 34.4$.

Problem: You have been employed as a mathematical modeller by a laboratory to help them establish the time of death of a murder victim. At $1pm$ when the pathologist arrived at the scene of the murder, she measured the temprature of the victim to be 15^oC. After 1 hour had elapsed she measured the temprature to be 12^oC.

1) Determine the time of death of the victim. Assume that the body temprature of the victim was 37^oC when he died and that the ambient temprature remained at 5^oC thoughour the period.

2) Now build a model given that the meterology department has establish that during the day temprature at that location was given by the function $T_a(t) = t - 2$, where t is the time measured in hours since sunrise at $6am$.

Solution: 1)

$$k = \frac{1}{t} \log \left(\frac{T_0 - T_a}{T - T_a} \right) \tag{17}$$

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$$k = \frac{1}{t} \log \left(\frac{T_0 - T_a}{T - T_a} \right) \tag{19}$$

$T_0 = 15^o C$, $T_a = 5^o C$, $T = 12^o C$, $t = 1$

Then , $k = \log \frac{10}{7}$

Solution: 1)

(21) $$k = \frac{1}{t} \log \left(\frac{T_0 - T_a}{T - T_a} \right)$$

$T_0 = 15^o C$, $T_a = 5^o C$, $T = 12^o C$, $t = 1$

Then , $k = \log \frac{10}{7}$

Now determine $t =?$ at $T = 37$

Solution: 1)

(23) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

$T_0 = 15^oC$, $T_a = 5^oC$, $T = 12^oC$, $t = 1$

Then , $k = \log\frac{10}{7}$

Now determine $t = ?$ at $T = 37$

Solution: 2)

$$T = t - 2 - \frac{1}{k} + Ce^{-kt}$$

Solution: 1)

(25) $$k = \frac{1}{t}\log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

$T_0 = 15^oC$, $T_a = 5^oC$, $T = 12^oC$, $t = 1$
Then , $k = \log\frac{10}{7}$
Now determine $t =?$ at $T = 37$

Solution: 2)

$$T = t - 2 - \frac{1}{k} + Ce^{-kt}$$

(26) $$T_7 = f(k, C) = 7 - 2 - \frac{1}{k} + Ce^{-7k} = 15$$
$$T_8 = g(k, C) = 8 - 2 - \frac{1}{k} + Ce^{-8k} = 12$$

These are nonlinear equation which you have to solve by Newton's method.