



*Math 2E03- Introduction to  
Modelling*

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## *Newton's Law of Cooling*

Someone determined that the rate of change of the temperature of an object (like a glass of milk) is proportional to the difference in temperature of the object and its surrounding. If we let the temperature of the milk be  $T$ , time  $t$ , and the environmental temperature  $T_a$ , then we have






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$$(2) \quad \frac{dT}{dt} = -k(T - T_a)$$



**Problem:** Reena, a fussy hot chocolate drinker, wants her water to be precisely  $85^{\circ}C$ , but she usually forgets and lets it boil. Having broken her thermometer, she asks you to calculate how long she should wait for it to cool from  $100^{\circ}C$  to  $85^{\circ}C$ . Assume that ambient temperature is constant. Can you solve her problem? if so, do so. If not, why not?

**Solution:**





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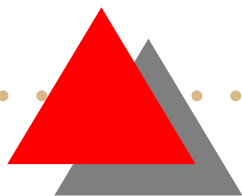
**Solution:**


Let's try for the solution of ODE for this model.

$$(3) \quad T = (T_0 - T_a)e^{-kt} + T_a$$

and the value of constant  $k$  can be found out by

$$(4) \quad k = \frac{1}{t} \log\left(\frac{T_0 - T_a}{T - T_a}\right)$$

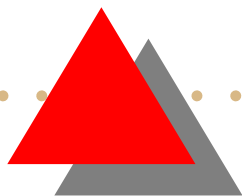





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**Solution:**

**Analysis of the result:** This is an important lesson - just because you can derive a theoretical ODE model and solve it, does not mean it will automatically be useful. The value of parameters and integration constants are determined by the specific system and conditions, and can in general only be determined empirically.





**Problem:** You have just returned home at 5pm on a hot day ( $35^{\circ}C$ ) after a shopping and are expecting some guests for dinner. Unfortunately you forgot to put the pop into refrigerator before you left. The thermometer in the refrigerator displays the temperature in the refrigerator to be  $5^{\circ}C$ . After 10 minutes in the refrigerator you observe that the pop has cooled to  $28^{\circ}C$ . Will the pop have cooled enough (say to  $10^{\circ}C$ ) in the remaining 80 minutes before your guests arrive? Or will you have to resort to putting the pop cans into the freezer (whose temperature is  $-2^{\circ}C$ ) and run the risk of forgetting them yet again -recall those time pop froze and burst their cans! What time should you set your alarm for in order to remind you when to take the pop out of freezer, should you go that route.





Solution :

(5)

$$k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$





Solution :

$$(7) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T = 28^\circ C \text{ at } t = 10, T_0 = 35^\circ C, T_a = 5^\circ C$$



Solution :

(9) 
$$k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$T = 28^\circ C$  at  $t = 10$ ,  $T_0 = 35^\circ C$ ,  $T_a = 5^\circ C$

$k \approx .0266$



Solution :

$$(11) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T = 28^\circ C \text{ at } t = 10, T_0 = 35^\circ C, T_a = 5^\circ C$$

$$k \approx .0266$$

We can now substitute in the time that the guest will arrive and see what the temprature of the pop will be

$$(12) \quad T = (T_0 - T_a)e^{-kt} + T_a$$



Solution :

$$(13) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T = 28^\circ C \text{ at } t = 10, T_0 = 35^\circ C, T_a = 5^\circ C$$

$$k \approx .0266$$

We can now substitute in the time that the guest will arrive and see what the temperature of the pop will be

$$(14) \quad T = (T_0 - T_a)e^{-kt} + T_a$$

$$t = 90, T_0 = 35^\circ C, T_a = 5^\circ C$$

$$T \approx 7.74$$



Solution :

$$(15) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T = 28^\circ C \text{ at } t = 10, T_0 = 35^\circ C, T_a = 5^\circ C$$

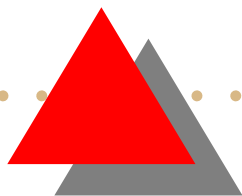
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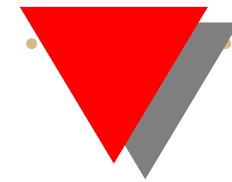
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$$(16) \quad T = (T_0 - T_a)e^{-kt} + T_a$$

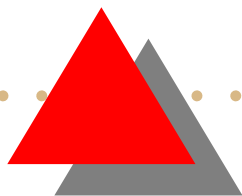
$$t = 90, T_0 = 35^\circ C, T_a = 5^\circ C$$


$T \approx 7.74$  It is less than  $10^\circ C$ , so we can leave in the fridge.





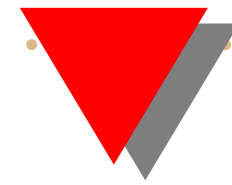
We can, then solve for the time  $t$  at which they will reach  $10^{\circ}C$  and set our alarm clock for this time to remind us to take out. Now the question is  $t = ?$





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$$T_0 = 28^{\circ}C, T_a = -2^{\circ}C, T = 10^{\circ}C \text{ so } t \approx 34.4.$$



**Problem:** You have been employed as a mathematical modeller by a laboratory to help them establish the time of death of a murder victim. At  $1pm$  when the pathologist arrived at the scene of the murder, she measured the temperature of the victim to be  $15^{\circ}C$ . After 1 hour had elapsed she measured the temperature to be  $12^{\circ}C$ .

1) Determine the time of death of the victim. Assume that the body temperature of the victim was  $37^{\circ}C$  when he died and that the ambient temperature remained at  $5^{\circ}C$  throughout the period.

2) Now build a model given that the meteorology department has established that during the day temperature at that location was given by the function  $T_a(t) = t - 2$ , where  $t$  is the time measured in hours since sunrise at  $6am$ .



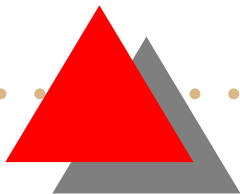




Solution: 1)

(17)

$$k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$



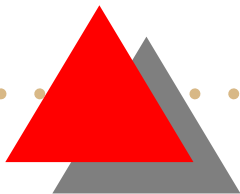


Solution: 1)

$$(19) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T_0 = 15^\circ C, T_a = 5^\circ C, T = 12^\circ C, t = 1$$

$$\text{Then, } k = \log \frac{10}{7}$$





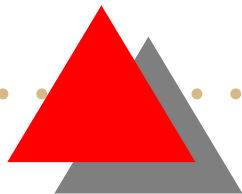
Solution: 1)

$$(21) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T_0 = 15^\circ C, T_a = 5^\circ C, T = 12^\circ C, t = 1$$

$$\text{Then, } k = \log \frac{10}{7}$$

Now determine  $t = ?$  at  $T = 37$





Solution: 1)

$$(23) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T_0 = 15^\circ C, T_a = 5^\circ C, T = 12^\circ C, t = 1$$

$$\text{Then, } k = \log \frac{10}{7}$$

Now determine  $t = ?$  at  $T = 37$

Solution: 2)

$$T = t - 2 - \frac{1}{k} + Ce^{-kt}$$

Solution: 1)

$$(25) \quad k = \frac{1}{t} \log \left( \frac{T_0 - T_a}{T - T_a} \right)$$

$$T_0 = 15^\circ C, T_a = 5^\circ C, T = 12^\circ C, t = 1$$

$$\text{Then , } k = \log \frac{10}{7}$$

Now determine  $t = ?$  at  $T = 37$

Solution: 2)

$$T = t - 2 - \frac{1}{k} + Ce^{-kt}$$

$$(26) \quad T_7 = f(k, C) = 7 - 2 - \frac{1}{k} + Ce^{-7k} = 15$$

$$T_8 = g(k, C) = 8 - 2 - \frac{1}{k} + Ce^{-8k} = 12$$

These are nonlinear equation which you have to solve by Newton's method.