Math 2E03- Introduction to Modelling

Instrutor– Dr. Mani Mehra

Department of Mathematics and Statistics
McMaster Univ.

You have seven dolls in your bedroom. You bring in 5 more another room, and take 2 of the original ones out into the other room. How many dolls do you have in your room now?.

Compartmental Models: Generally speaking, we have compartments, or boxes(lakes, tanks etc.) with a volume V (which may or may not be constant) and some kind of flow in to the box and some kind of the flow out of the box. We generally keep track of certain objects or species (which may be particles of pollution in the lake), keeping a simple account of how much is going in and how much is going out. If x is our object of interest, then we have

$$\Delta x = x_{in} - x_{out} \tag{1}$$

We are always interested in the amount of stuff- this may be measured directly or indirectly. The differential equation, though, must represent the change in the amount of stuff even if concentration values are used at times. We begin by looking at the flow of the particles in a medium: if the input has a concentration k of the particulate x, and is flowing at the rate of r_{in} (in voulme per time), then the input of x per unit time is kr_{in}. At any given time we will have $x(t)$ of x in the compartment, so its concentration is $\frac{x(t)}{V}$. If the compartment is well stirred (i.e concentartion is constant throughout the comaprtment) and the outflow is at rate of r_{out}, then the amount of x lost is $\frac{x_t}{V}r_{out}$. The above equation then becomes

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(3) $$\frac{dx}{dt} = kr_{in} - \frac{x}{V}r_{out}$$

for constant volumes.

If the volume is not constant, then

(4)
$$\frac{dV}{dt} = r_{in} - r_{out}$$

Solution is

$$V(t) = V_0 + (r_{in} - r_{out})t$$

and the above equation becomes

(5)
$$\frac{dx}{dt} = kr_{in} - \frac{xr_{out}}{V_0 + (r_{in} - r_{out})t}$$

Problem : Consider a large tank holding $1000L$ os water into which a salt solution begins to flow at a constant rate of $6Lmin^{-1}$. The solution inside the tank is kept well stirred and is also flowing out of the tank at a rate of $6Lmin^{-1}$.

1) If the concentratio of the salt entering in the solution is $2gL^{-1}$, determine how long it takes for the concentartion of the salt in the tank to reach $1gL^{-1}$.

2) How does the model change if the salt solution leaves the tank at only $5Lmin^{-1}$ instead of $6Lmin^{-1}$

Solution:

$$\frac{dA}{dt} = r_1 c_1 - r_2 \frac{A}{V}$$

Solution of this differential equation is

$$\frac{A}{V} = c_1 - B' e^{-\frac{rt}{V}}$$

First we have to determine the value of constant B'.
Using the $\frac{A}{V} = 0$ at $t = 0$.
We get $B' = 2gL^{-1}$
and using the formula

$$t = -\frac{V}{r} log(\frac{c1 - \frac{A}{V}}{B'})$$

we get $t \approx 115.5 min$