Math 2E03- Introduction to Modelling

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Close Encounters of the Inter, Interaspecies Equations

Model: Chances of encounter in Differential Equations If your town is the victim of the dread epidemic ‘cooties’, the chances of your catching cooties is propotional to the probability that you will bump into a carrier of the disease. So the probability of catching the disease is propotional to the probability that you will be at a given place at a given time multiplied by the probability that a carrier will also be in that given place at given time. If we represent the probability of an encounter simply by the susceptible individuals (S) time the number of infected individuals (I)

then we can represent this mathematically as

$$\frac{dI}{dt} \propto SI \qquad (1)$$
$$\frac{dS}{dt} \propto -SI$$

where I increses due to the encounters and S decresses by the same amount . In the same way other variation and conditions (like death or vaccinations), we can construct a system of ODEs to represent the overall system.

Model: Epidemic-Type Models There are number of varations amongst epidemic problems.

SIS Epidemics: which stands for Susceptible $\rightarrow$ Infected $\rightarrow$ Susceptible epedemics, such diseases as the common cold which generally cause a person to be sick for a time and then recover.

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Let the total number of people N in the town stays constant. Since everyone is either susceptible or infectious, $S + I = N$. So we can write our system of ODEs as

$$\frac{dS}{dt} = (b - aS)I$$
$$\frac{dI}{dt} = (a(N - I) - b)I = (A - aI)b \text{ where } A = aN - b \quad (7)$$

Now

$$\frac{dI}{(A-aI)b} = dt$$

(8)

$$\frac{1}{A}\left(\frac{1}{I}+\frac{a}{A-aI}\right)dI = dt$$

integrating both side

$$\log I - \log(A-aI) = At + Ac \text{ where c is integration constant}$$

(9)

$$\frac{I}{(A-aI)} = e^{At}B \text{ where } B = e^{Ac}$$

So the solution of bottom ODE is $I = \frac{ABe^{At}}{1+aBe^{At}}$

Now we solve for integration constant -at $t = 0$, $I = I_0$. Using this we get $B = \frac{I_0}{(aN-b)-aI_0}$.

Now substituting the value of B in the solution we get

$$I = \frac{(aN-b)I_0 e^{(aN-b)t}}{(aN-b) + aI_0[e^{(aN-b)t} - 1]} \tag{10}$$

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$$I = \frac{(aN-b)I_0 e^{(aN-b)t}}{(aN-b)+aI_0[e^{(aN-b)t}-1]} \tag{13}$$

$$S = N - \frac{(aN-b)I_0 e^{(aN-b)t}}{(aN-b)+aI_0[e^{(aN-b)t}-1]} \tag{14}$$

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Stability analysis: The SIS epedemic model is

$$\frac{dS}{dt} = bI - aSI$$
$$\frac{dI}{dt} = aSI - bI \tag{18}$$

If we devide the top equation by the bottom one we eliminate the time variable and end up with an easily solvable ODE

$$\frac{dI}{dS} = \frac{(aS-b)I}{-(aS-b)I} \tag{19}$$

The solution if $I = -S + N$.

If we devide the top equation by the bottom one we eliminate the time variable and end up with an easily solvable ODE

$$\frac{dI}{dS} = \frac{(aS-b)I}{-(aS-b)I} \tag{21}$$

The solution if $I = -S + N$.

Now we are looking for equilibrium points. Thus we check when both derivatives will be zero.

$$\begin{aligned} \frac{dS}{dt} &= (b-aS)I = 0 \text{ when } S = \frac{b}{a} \text{ or when } I = 0 \\ \frac{dI}{dt} &= (aS-b)I = 0 \text{ when } S = \frac{b}{a} \text{ or when } I = 0 \end{aligned} \tag{22}$$

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$$\frac{dI}{dS} = \frac{(aS-b)I}{-(aS-b)I} \tag{23}$$

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The first case assumes that $\rho = \frac{b}{a} < N$ and in second case $\rho > N$. In case 1 we have the equilibrium points $(N, 0)$ and $(\rho, N-\rho)$. In case 2 we have only equlibrium point $(N, 0)$.

Looking at case 1 at the $(\rho, N-\rho)$ equilibrium point, we see that if it is stable then to the right of that point the value of S should be decresing $(\dot{S} = \frac{dS}{dt} < 0)$ towards the point. Same arguement we could have made for I.

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$$\begin{aligned} \frac{dS}{dt} &= (b - aS)(N - S) \\ &= aS^2 - (Na + b)S + bN \end{aligned} \tag{26}$$

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$$\frac{dS}{dt} = (b - aS)(N - S) \tag{27}$$
$$= aS^2 - (Na + b)S + bN$$

Now by plotting the graph between $\dot{S}$ and S we can see that for case 1 $(\rho, N-\rho)$ is stable while $(N, 0)$ is not, while for case 2 $(N, 0)$ appears to be stable.