Math 2E03- Introduction to Modelling

Instructor– Dr. Mani Mehra

Department of Mathematics and Statistics
McMaster Univ.

Introducing Perturbation: As you will recall the method for determining the stability of an equilibrium point $(\bar{S}, \bar{I})$ is to introduce small perturbation and see what happens. These perturbation will not be constants-they can change with time. We are adding ϵ to $\bar{S}$, δ to $\bar{I}$, and substitute these in to ODEs

$$S = \bar{S} + \epsilon$$

$$I = \bar{I} + \delta$$

$$\dot{S} = \dot{\bar{S}} + \dot{\epsilon}$$

$$\dot{S} = [b - a(\bar{S} + \epsilon)](\bar{I} + \delta)$$

$$\dot{\epsilon} = (b - a\bar{S})\bar{I} + (-a\bar{I})\epsilon + (b - a\bar{S})\delta - a\epsilon\delta$$

after ignoring some terms

$$\dot{\epsilon} \approx (-a\bar{I})\epsilon + (b - a\bar{S})\delta$$

Similarly we can drive

$$\dot{\delta} \approx (a\bar{I})\epsilon + (a\bar{S} - b)\delta$$

We can write it in a matrix form

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -a\bar{I} & b - a\bar{S} \\ a\bar{I} & a\bar{S} - b \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$$

Now how to solve this system of ODEs?. Looking at $(\bar{S}, \bar{I}) = (N, 0)$ for both cases

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$$

We will solve this system of ODEs by reducing it into eigenvalues problem.

Let $\epsilon = Ae^{\lambda t}$ and $\delta = Be^{\lambda t}$. If we let $x(t) = \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$ and $v = \begin{bmatrix} A \\ B \end{bmatrix}$, then $x(t) = e^{\lambda t} v$ so that we can rewrite the above matrix equation as follows

(1) $$\dot{x(t)} = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} x(t)$$

(2) $$\lambda e^{\lambda t} v = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} e^{\lambda t} v$$

Since $v \neq 0$ so we have the following eigenvalue problem:

(3) $$\lambda v = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} v$$

.

we get two eigenvalues $\lambda = 0, aN - b$ Thus we have the solution correspond to $\lambda = 0$

$$x(t) = e^{0} v$$

(4)
$$= n \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

where n is any real number. Similarly solution correspond to $\lambda = aN - b$ is

(5)
$$x(t) = e^{(aN-b)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For case 1 $aN - b > 0$, so $e^{(aN-b)t}$ increases as time increases-we get two eigenvalues $\lambda = 0, aN - b$ Thus we have the solution correspond to $\lambda = 0$

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$$x(t) = e^{(aN-b)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For case 1 $aN - b > 0$, so $e^{(aN-b)t}$ increases as time increases-thus $(N, 0)$ is an unstable equilibrium point.

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SIR Epidemics

SIR stands for Susceptible$\rightarrow$Infected$\rightarrow$Removal epidemics. The term removal is a general one which allows for infected individuals to be no longer infected, yet not susceptible either.

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$$\begin{aligned} \frac{dS}{dt} &= -aSI \\ \frac{dI}{dt} &= aSI - bI \qquad (9) \\ \frac{dR}{dt} &= bI \end{aligned}$$

We are assuming that the rate of recovery/death is proportional to number of sick people.

Vaccinations/Cures in epidemic modelling

People can go directly from the susceptible to the Immune category through the adminstration of a vaccine. Sometime the number of vaccinations given to be proportional to the number of susceptible, Infected (sick) or recovered people. But generally a certain number of shots would be given per day or time period. Let us suppose some parameter c that represents the number of vaccinations given per time period. Then the model will look like

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$$\begin{aligned} \frac{dS}{dt} &= -aSI - c \\ \frac{dI}{dt} &= aSI - bI \\ \frac{dR}{dt} &= bI + c \end{aligned} \tag{11}$$

Chemical Reactions

Model of reaction rates:

Consider the following equation for the oxidation of iron

$$3Fe + 2O_2 \rightarrow Fe_3O_4 \quad (12)$$

One would expect the increase in the amount of iron oxide to be proportional to the decreases in oxygen and in iron. It is important to note that 2 moles of oxygen and 3 of iron are needed, so for every decraeses in oxygen of 2 moles there will be an incraese in only 1 mole of iron oxide. Thus the decease in a reactant must be devided by its stoiciometric coeffcient:

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$$\frac{d[Fe_3O_4]}{dt} = -\frac{1}{2}\frac{d[O_2]}{dt} = -\frac{1}{3}\frac{d[Fe]}{dt} \tag{15}$$

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$$r_1R_1 + r_2R_2 + \cdots + r_nR_n \rightarrow p_1P_1 + p_2P_2 + \cdots + p_kP_k$$

$$\Longrightarrow \left(\frac{1}{p_i}\right)\frac{d[P_i]}{dt} = -\left(\frac{1}{r_j}\right)\frac{d[R_j]}{dt} \tag{17}$$