




*Math 2E03- Introduction to
Modelling*

Instructor– Dr. Mani Mehra

Department of Mathematics and Statistics
McMaster Univ.





Introducing Perturbation: As you will recall the method for determining the stability of an equilibrium point (\bar{S}, \bar{I}) is to introduce small perturbation and see what happens. These perturbation will not be constants-they can change with time. We are adding ϵ to \bar{S} , δ to \bar{I} , and substitute these in to ODEs

$$S = \bar{S} + \epsilon$$

$$I = \bar{I} + \delta$$

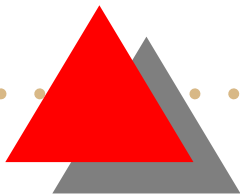
$$\dot{S} = \dot{\bar{S}} + \dot{\epsilon}$$

$$\dot{S} = [b - a(\bar{S} + \epsilon)](\bar{I} + \delta)$$

$$\dot{\epsilon} = (b - a\bar{S})\bar{I} + (-a\bar{I})\epsilon + (b - a\bar{S})\delta - a\epsilon\delta$$

after ignoring some terms

$$\dot{\epsilon} \approx (-a\bar{I})\epsilon + (b - a\bar{S})\delta$$





Similarly we can derive

$$\dot{\delta} \approx (a\bar{I})\epsilon + (a\bar{S} - b)\delta$$

We can write it in a matrix form

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} -a\bar{I} & b - a\bar{S} \\ a\bar{I} & a\bar{S} - b \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$$

Now how to solve this system of ODEs?. Looking at $(\bar{S}, \bar{I}) = (N, 0)$ for both cases

$$\begin{bmatrix} \dot{\epsilon} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$$

We will solve this system of ODEs by reducing it into eigenvalues problem.





Let $\epsilon = Ae^{\lambda t}$ and $\delta = Be^{\lambda t}$. If we let $x(t) = \begin{bmatrix} \epsilon \\ \delta \end{bmatrix}$ and $v = \begin{bmatrix} A \\ B \end{bmatrix}$, then

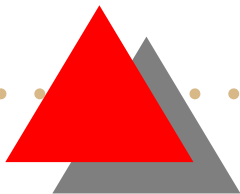
$x(t) = e^{\lambda t}v$ so that we can rewrite the above matrix equation as follows


$$(1) \quad \dot{x}(t) = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} x(t)$$

$$(2) \quad \lambda e^{\lambda t}v = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} e^{\lambda t}v$$

Since $v \neq 0$ so we have the following eigenvalue problem:

$$(3) \quad \lambda v = \begin{bmatrix} 0 & b - aN \\ 0 & aN - b \end{bmatrix} v$$





we get two eigenvalues $\lambda = 0, aN - b$ Thus we have the solution correspond to $\lambda = 0$


$$(4) \quad \begin{aligned} x(t) &= e^0 v \\ &= n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

where n is any real number. Similarly solution correspond to $\lambda = aN - b$ is

$$(5) \quad x(t) = e^{(aN-b)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For case 1 $aN - b > 0$, so $e^{(aN-b)t}$ increases as time increases-






we get two eigenvalues $\lambda = 0, aN - b$ Thus we have the solution correspond to $\lambda = 0$

$$(6) \quad \begin{aligned} x(t) &= e^0 v \\ &= n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

where n is any real number. Similarly solution correspond to $\lambda = aN - b$ is


$$(7) \quad x(t) = e^{(aN-b)t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

For case 1 $aN - b > 0$, so $e^{(aN-b)t}$ increases as time increases-thus $(N, 0)$ is an unstable equilibrium point.

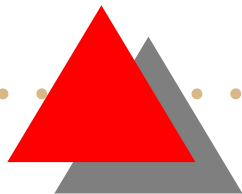



For case 2 $aN - b < 0$, so $e^{(aN-b)t}$ decreases as time increases





For case 2 $aN - b < 0$, so $e^{(aN-b)t}$ decreases as time increases so point $(N, 0)$ is stable equilibrium point.

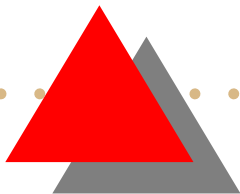





For case 2 $aN - b < 0$, so $e^{(aN-b)t}$ decreases as time increases so point $(N, 0)$ is stable equilibrium point. For case 1, now we will have the equilibrium point $(\rho, N - \rho)$ to look at. Now we get two eigenvalues

$\lambda = 0, b - aN$. So solution correspond to $\lambda = 0$ is $x(t) = n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So

solution correspond to $\lambda = b - aN$ is $x(t) = e^{(b-aN)t} n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.





For case 2 $aN - b < 0$, so $e^{(aN-b)t}$ decreases as time increases so point $(N, 0)$ is stable equilibrium point. For case 1, now we will have the equilibrium point $(\rho, N - \rho)$ to look at. Now we get two eigenvalues

$\lambda = 0, b - aN$. So solution correspond to $\lambda = 0$ is $x(t) = n \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. So

solution correspond to $\lambda = b - aN$ is $x(t) = e^{(b-aN)t} n \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus this point is a stable equilibrium point.



SIR Epidemics

SIR stands for Susceptible→Infected→Removal epidemics. The term removal is a general one which allows for infected individuals to be no longer infected, yet not susceptible either.



SIR Epidemics

SIR stands for Susceptible→Infected→Removal epidemics. The term removal is a general one which allows for infected individuals to be no longer infected, yet not susceptible either.

$$(9) \quad \begin{aligned} \frac{dS}{dt} &= -aSI \\ \frac{dI}{dt} &= aSI - bI \\ \frac{dR}{dt} &= bI \end{aligned}$$

We are assuming that the rate of recovery/death is proportional to number of sick people.



Vaccinations/Cures in epidemic modelling

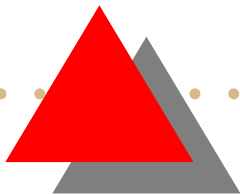
People can go directly from the susceptible to the Immune category through the administration of a vaccine. Sometime the number of vaccinations given to be proportional to the number of susceptible, Infected (sick) or recovered people. But generally a certain number of shots would be given per day or time period. Let us suppose some parameter c that represents the number of vaccinations given per time period. Then the model will look like



Vaccinations/Cures in epidemic modelling

People can go directly from the susceptible to the Immune category through the administration of a vaccine. Sometime the number of vaccinations given to be proportional to the number of susceptible, Infected (sick) or recovered people. But generally a certain number of shots would be given per day or time period. Let us suppose some parameter c that represents the number of vaccinations given per time period. Then the model will look like

$$(11) \quad \begin{aligned} \frac{dS}{dt} &= -aSI - c \\ \frac{dI}{dt} &= aSI - bI \\ \frac{dR}{dt} &= bI + c \end{aligned}$$





Chemical Reactions

Model of reaction rates:

Consider the following equation for the oxidation of iron



One would expect the increase in the amount of iron oxide to be proportional to the decreases in oxygen and in iron. It is important to note that 2 moles of oxygen and 3 of iron are needed, so for every decrease in oxygen of 2 moles there will be an increase in only 1 mole of iron oxide. Thus the decrease in a reactant must be divided by its stoichiometric coefficient:



Chemical Reactions

Model of reaction rates:

Consider the following equation for the oxidation of iron

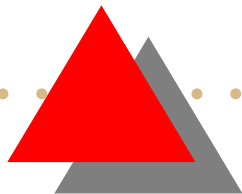


One would expect the increase in the amount of iron oxide to be proportional to the decreases in oxygen and in iron. It is important to note that 2 moles of oxygen and 3 of iron are needed, so for every decrease in oxygen of 2 moles there will be an increase in only 1 mole of iron oxide. Thus the decrease in a reactant must be divided by its stoichiometric coefficient:

$$(15) \quad \frac{d[Fe_3O_4]}{dt} = -\frac{1}{2} \frac{d[O_2]}{dt} = -\frac{1}{3} \frac{d[Fe]}{dt}$$



In more general terms, for reactants R , products P and their coefficients r and p





In more general terms, for reactants R , products P and their coefficients r and p

$$(17) \quad r_1 R_1 + r_2 R_2 + \cdots + r_n R_n \rightarrow p_1 P_1 + p_2 P_2 + \cdots + p_k P_k$$
$$\implies \left(\frac{1}{p_i} \right) \frac{d[P_i]}{dt} = - \left(\frac{1}{r_j} \right) \frac{d[R_j]}{dt}$$

