Math 2E03- Introduction to Modelling

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We differentiate the first ODE with respect to time and then substitute the second ODE in to the result

$$\ddot{\epsilon} = \frac{bc}{d}\dot{\delta} \qquad (1)$$

$$\ddot{\epsilon} = \frac{bc}{d}(-\frac{ad}{b}\epsilon)$$

So

$$\ddot{\epsilon} + ac\epsilon = 0$$

Thus the solution

$$\epsilon = A\cos(\sqrt{(}ac)t) + B\sin(\sqrt{(}ac)t)$$

Thus ϵ oscillates sinusoidally with time about the equilibrium.

We now turn our attentions to the other equilibrium point $(0,0)$

$$x = 0 + \epsilon$$
$$y = 0 + \delta \qquad (2)$$
$$\dot{x} = \dot{\epsilon}$$
$$\dot{y} = \dot{\delta}$$

So

$$\dot{\epsilon} = (-a + b\delta)\epsilon \qquad (3)$$
$$\dot{\epsilon} = -a\epsilon$$

Similarly $\dot{\delta} = c\delta$ which is an extremely easy system to solve directly. The first ODE gives us $\epsilon = Ae^{-at}$, and the second gives $\delta = Be^{ct}$ where A and B are constants of integration which is this case represent the pop. at $t = 0$ (ϵ_0 and δ_0).

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Prey Overcrowding: If all the foxes died out in our previous model, the rabbit pop. would grow unchecked forever. Is this realistic? No, of course not. After a while there would not be enough food to support all the rabbits and some would inevitably die of starvation. We include a $-ey^2$ term in the prey ODE to represent the negative effect other rabbits can have on particular rabbit-they may beat her to a source of food, or she may become injured in a fight over shelter. This term can also represent the spread of disease amongst the rabbits-as the pop. increases the rabbits who must go without food or shelter become sickly.

(4)
$$\dot{x} = (-a + by)x$$
$$\dot{y} = (c - dx - ey)y$$

Hunting: Each and every year people go out hunting various animals, be it for food or for sport. The government regulates such activities, trying to limit the number of certain species killed to a certain percentage of the estimated population.

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$$\begin{aligned} \dot{x} &= (-a + by)x - H_x x \\ \dot{y} &= (c - dx)y - H_y y \end{aligned} \tag{7}$$

Problem: Build a model for the evolution of a population of algae in a bay that feed off the phosphates from detergents that we assume run into the bay at a constant rate.

- To start with assume that the pop. of algae is so small that they do not effect the concentration of nutrients. Build a model of this situation and solve it. What happens to the size of algae pop.?
- Now assume that the pop. of algae has reached a level at which they have to compete among themselves for the resources comprising the phosphates. Build this situation into the model and solve it to determine the outcome.

- With no other factors to consider and a plentiful food source, the algae (V) will simply grow at their optimum growth rate r.

$$\frac{dV}{dt} = rV \tag{8}$$

Solution is $V = Ae^{rt}$. The pop. grows exponentially.

- With no other factors to consider and a plentiful food source, the algae (V) will simply grow at their optimum growth rate r.

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- We must include a detrimental interaction amongst the algae

$$\dot{V} = rV - gV^2$$

$$dV\left[\frac{1}{V} + \frac{g}{r - gV}\right] = rdt$$

(11) Integrating both side

$$\log V - \log(r - gV) = rt + C$$

$$\frac{V}{r - gV} = Ae^{rt}$$

Using at $t = 0$ $V = V_0$ the solution is $V = \frac{r}{g + (\frac{V}{V_0} - g)e^{-rt}}$