Math 2E03- Introduction to Modelling

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Age structure: the simplest case:

Many species of interest do'not have the simple life history that allows us to blindly use the models we have just developed. Females of snapping turtles do not become sexually mature until they are more than 5 years old, and continue to lay eggs essentially throughout their life, which may be as long as 100 years. How should we understand the dynamics of this species.?.....

Age structure: the simplest case:

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Befor considering forces that prevent populations from growing expontially, we will consider whether our prediction of expontial growth holds in a slightly more complex setting where we include age structure. We will study a hypothetical organism that lives for 2 years, potentially reproducing either at age 1 or at age 2. We will call an organism 0 years old during its first year of life and 1 year old during its second year. We will assume that all individuals die before they reach their third year.

The parameters and variables we need to describe this are as follows.

- m_0 is the mean number of offspring of a 0 year old the following year.
- m_1 is the mean number of offspring of a 1 year old the following year.
- S_0 is the probability that a 0 year old survives to become a 1 year old.
- $n_0(t)$ is the number of 0 year olds at time t.
- $n_1(t)$ is the number of 1 year olds at time t.
- $N(t) = n_0(t) + n_1(t)$ is the total number of organisms at time t.

We first describe the model in words, and then in equations

- $n_0(t+1) =$ number of offspring of 0 year olds in year $t +$ number of offspring of 1 year olds in year t.
- $n_1(t+1) =$ number of 0 year olds in year t times the probability of survival from 0 to 1.

Translating this into equations, we find that

$$\begin{aligned} n_0(t+1) &= n_0(t)m_0 + n_1(t)m_1 \\ n_1(t+1) &= n_0(t)S_0 \end{aligned} \tag{1}$$

Does a population grow exponentially even we have considered age structure model??

Problem 1: What happens if in year 0 we have 10 zero year olds and no one year olds... $n_0 = 10$, $n_1 = 0$ are the starting sizes in each year class. Using equations (1) we see the next year

$$n_0(1) = 10m_0$$
$$n_1(1) = 10S_0 \tag{2}$$

Problem 2: If in year 0 we have no zero year olds and 10 one year olds , so that $n_0(0) = 0$ and $n_1(0) = 10$. Using equation (1) we see the next year

$$n_0(1) = 10m_1$$
$$n_1(1) = 0 \tag{3}$$

There are very different populations, so we conclude that no single number represents a growth rate. How could we get a specific, constant rate?

Let us assume that the ratio of 0 years olds to 1 years olds remain the same i.e $n_1(t) = cn_0(t)$, where c is the ratio of one year olds to zero years old. Substituing in to model (1), we find

$$
\begin{aligned}
n_0(t+1) &= n_0(t)m_0 + cn_0(t)m_1 \\
cn_0(t+1) &= n_0(t)S_0
\end{aligned}
\tag{4}
$$

Now eliminating $n_0(t+1)$ and $n_0(t)$, we get

$$
c^2 m_1 + cm_0 - S_0 = 0 \tag{5}
$$

$$
c = \frac{-m_0 \pm \sqrt{(m_0^2 + 4S_0 m_1)}}{2m_1} \tag{6}
$$

From equation (1) we see that the growth rate of the population is S_0/c, so for the positive value of c, S_0/c is candidate for the growth rate of population.

A much more convenient notation , essential for extending the models to include more age classes, is to use matrices. We can write the model (1) using matrices as

$$\begin{pmatrix} m_0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} n_0(t) \\ n_1(t) \end{pmatrix} = \begin{pmatrix} n_0(t+1) \\ n_1(t+1) \end{pmatrix}$$

We will assume that the ratio of 0 years olds to 1 year olds remains constant and the population grows at the rate λ per year. We call this ratio of individuals at different ages that remains constant a stable age distribution. Thus, the stable age distribution is not unique-if 10 zero year olds and 5 one years old is a stable age distribution, so is 20 zero years olds and 10 one years olds.

We now express the assumption that the pop. is in a stable age distribution in matrix and vector notation. The idea is that in every year the ratio of 0 years old to 1 years olds remains constatnt, but that the number in ecah age class grows at the rate λ each year. We write

$$\begin{pmatrix} m_0 & m_1 \\ S_0 & 0 \end{pmatrix} \begin{pmatrix} n_0(t) \\ n_1(t) \end{pmatrix} = \begin{pmatrix} \lambda n_0 \\ \lambda n_1 \end{pmatrix} \tag{7}$$

In honour of P. H. Leslie, who was first one to describe the use of matrices to dexcribe population dynamics with age structure, we will call the matrix in this equation Leslie matrix and denote it by L. Let N be the vector of population sizes $N = (n_0 \; n_1)'$, then we can write model for the stable age distribution in more compact form

$$LN = \lambda N \tag{8}$$