Math 2E03- Introduction to Modelling

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A survey of models containing PDEs

When creating our ODEs previously we have been only inserted in systems which changes with respect to one variable in particular changes with respect to time. However many systems insist on behaving in a much more complicated manner. For instnace, if a drop of dye is placed in a pail the concentration at some point will depend both on the point's position relative to the placement of dye and on the time elapsed since the dye was introduced. The concentration at that point changes with space and time.

In response we turn to PDEs, which allow us to identify the different variables that a system may be dependent on and set up our models accordingly.

What is a partial differential equation The simplest form of partial differential equations (PDE) involving a suitably differentiable unknown function $u(x, y)$ of the two independent variables x and y is an equation that relates x, y, u and some partial derivatives of u with respect to x and y. The order of the PDE is the order of the highest partial derivative of u that occurs in the equation, so a general first order PDE for the function $u(x, y)$ is of the form

$$F(x, y, u, u_x, u_y) = 0 \tag{1}$$

where F is an arbitrary function of its arguement.

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$$F(x, y, u, u_x, u_y) = 0 \tag{3}$$

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More generally, a first order PDE for a function $(u(x_1, \cdots, x_n)$ of the n independent variables $x_1, \cdots, x_n$ is an equation of the form

$$G(x_1, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}) = 0 \tag{4}$$

where G is an arbitrary function of its arguements and $u_{x_i} = \frac{\partial u}{\partial x_i}$, for $i = 1, 2, \cdots, n$

Conservation law

It states simply that mass can either be created nor destroyed. One of the implications of this law is that in modelling the spread of of particles (like pollution in a lake or cars on a road) we must create models which do not allow for the spontaneous creation or disappearance of any of our particles. The mathematical form of this law can be used as the first step in creating models to account for the movement of such particles.

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(a) Deriving the Conservation law: We will represent the density of cars at a particular time and place as $U(x, t)$. The flow of cars F will be related in some manner to the density of the cars. The total flow represents the change in total number of cars with respect to time. Another way to calculate the total flow is just as the mass of an object is calculated by summing the density of small sections of it.

The total number of cars on stretch of road can be calculated by summing the density of these small portions over the total interval from x to $x + \Delta x$ is $\int_x^{x+\Delta x} U(x,t)dx$. Now change in this number with respect to time $\frac{d}{dt}\int_x^{x+\Delta x} U(x,t)dx$ is the change in total number of cars with respect to time....the total flow. Thus we can equate the two expressions for the total flow.

$$
\begin{aligned}
&\frac{d}{dt}\int_x^{x+\Delta x} U(x,t)dx = F[U(x,t)] - F(U(x+\Delta x,t)] \\
&\int_x^{x+\Delta x} \frac{\partial U(x,t)}{\partial t} = F[U(x,t)] - F(U(x+\Delta x,t)]
\end{aligned} \tag{5}
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\end{aligned} \tag{7}
$$

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$$
\begin{aligned}
\frac{\partial U(x,t)}{\partial t}\Delta x &\approx F[U(x,t)] - F(U(x+\Delta x,t)] \\
\frac{\partial U(x,t)}{\partial t} &= -\frac{\partial F[U(x,t)]}{\partial x}
\end{aligned} \tag{8}
$$

We have now derived the conservation law

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \tag{9}$$

Since F is a function of U

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial U}\frac{\partial U}{\partial x} = 0 \tag{10}$$

if $\frac{\partial U}{\partial t} > 0$ the density of car is increasing ans so the flow of cars past some point x should be decreasing.

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(b) The Differential Derivation of the conservation Law

We use the same strip of road x long and essentially the same arguemnets equating the two total flow equations to one another. The changing density $\frac{\partial U}{\partial t}$, when multiplied by the total length Δx should approximately represent the change in number of car over this interval. Choosing a point at the middle and calling it x, we can see that the edges represent the points $x - \frac{\Delta x}{2}$ and $x + \frac{\Delta x}{2}$

Thus the flow in minus the flow out is $F[U(x - \frac{\Delta x}{2}, t)] - F(U(x + \frac{\Delta x}{2}, t)$

$$\frac{\partial U(x,t)}{\partial t}\Delta x \approx F[U(x - \frac{\Delta x}{2}, t)] - F(U(x + \frac{\Delta x}{2}, t)]$$

(13)

$$\frac{\partial U}{\partial t} \approx -\frac{\partial F}{\partial x}$$

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\frac{\partial U(x,t)}{\partial t}\Delta x &\approx F[U(x-\frac{\Delta x}{2},t)]-F(U(x+\frac{\Delta x}{2},t)] \\
\frac{\partial U}{\partial t} &\approx -\frac{\partial F}{\partial x}
\end{aligned}
\tag{15}
$$

(c) The integral derivation of the conservation law

Once again we will consider a certain stretch of road -this time from $x=a$ to $x=b$. The sensity U at some time t_2 minus U at t_1, integarted over every interval dx from a to b will represent the change in the total number of cars on this strech of road from t_1 to t_2.

Similarly, the flow at a minus the flow at b, integrated over every time interval dt from t_1 to t_2 will also represent the change in the total number of cars on this strech of road from t_1 to t_2. Therefore

$$
\int_{1}^{b}[U(x,t_2)-U(x,t_1)]dx=\int_{t_1}^{t_2}F[U(a,t)]-F[U(b,t)]dt
\tag{16}
$$

We can now rewrite our equation

$$\int_a^b \int_{t_1}^{t_2} \frac{\partial U}{\partial t} dt dx = \int_{t_1}^{t_2} \int_a^b -\frac{\partial F}{\partial x} dx dt \tag{17}$$

$$\implies \frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{18}$$