# *Math 2E03- Introduction to Modelling*

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**Special examples of $F(U)$:** What can affect particles flowing through a medium?. Convection (or advection) is the term for movement of particles in a medium that has its own internal velocity. For example, pollen blown by the wind is moving through the air, driven by convection. Diffusion is the term given to ‘random spreading’ like that seen when we drop dye into water, for instance. It is a result of Brownian Motion( the vibration of the molecules of the medium ), which in and of itself can be affected by serveral factors including temperature. Let’s look at what the representation of flow is for these kinds of motion.

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Diffuison:

$$F(U) = -D\frac{\partial U}{\partial x}$$

$$\Longrightarrow \frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2}$$

**Convection/Diffusion** while only one or the other may be prevalent in a given system, convection and diffusion are often at work simultaneously-in the example given for the convection equation (the pollen blowing in the wind) diffusion is also at work, accounting in particular for any lateral spreading of the pollen with respect to the wind direction. We simply include both convection and diffusion terms in the equation.

$$\frac{\partial U}{\partial t} + c\frac{\partial U}{\partial x} = D\frac{\partial^2 U}{\partial x^2}$$

## *Convection equation*

Since we think the solution to the convection equation may be in the form of a wave, we will need to borrow the idea of galilean transformation in order to gain more insight into our equation.

Imagine Fiona and Amber sitting on a pier. Fiona is on a surfboard waiting for the next big wave. She measures distances cordinates with primes $x^{'}$ while Amber, stationary on the pier, measures distances without primes $x$.

A wave rolls in at a speed $c$ and Fiona rides it in the $x$ direction. At the time $t$ she will have a travelled a distance $ct$ and her origin will be that far away from Amber's. Fiona would measure Amber as being at $-ct$ since she travelled in the positive $x$ direction. If Fiona spots some people fishing at a distance $x^{'}$ away from her, what does Amber measure their distance as?.

Thus the total distance as measured by Amber is $x = ct + x^{'}$.
A Galilean Transformation takes one (in classical physics) from one frame of reference to another frame of reference moving (with a constant velocity) relative to the first, perhaps also with a different definition of the origin of time and position. $x -> x^{'} = x + vt - x_0^{'}$
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$$U(x,t) = \bar{U}(x - ct)$$

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Solution to the convection equation We wish to make the substituion $x - ct$ into $U$. We thus have the following.

$$s = x - ct$$

$$\frac{\partial s}{\partial x} = 1$$

$$\frac{\partial s}{\partial t} = -c$$

$$U(x,t) = \bar{U}(s)$$
$$\frac{\partial \bar{U}}{\partial x} = \bar{U}'$$
$$\frac{\partial \bar{U}}{\partial t} = -c\bar{U}'$$

which we can substitute into our convection equation

$$\frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{U}}{\partial t} = 0$$

The PDE was satisfied without any kind of restriction on the function itself except that it had to represent a travelling wave of some sort. This means that any wave like function can be a solution, as long as it is a function of $(x - ct)$. For example, try $sin(x - ct)$

How do we interpret such a solution?. We think back to Fiona cruising along her cupboard. She should stay at whatever height she was at when she caught the wave and so her height is constant. Normally to an observer the height of the wave would be dependent upon time and distance but by substituting $x - ct$ we have switched to Fiona's coordinates and are riding the wave with her. To her the wave looks perfectly motionless since she is riding along with it at the same speed and is always at the same height. The height of wave seems to be constant as well. If $U$ represents the height of the wave, then $U$ appear to be constant. This is only true if

$x - ct = A$ using $x = x_0$ at $t = 0$ we have $x = ct + x_0$

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As time goes by this entire wave moves along $x = ct + x_0$ line. So we can create our general solution of convection equation is

$$U(x,t) = U_0(x - F^{'}(U)t)$$

## *Shock Waves*

An interesting phenomena that can be explained nicely with our model above is the cause of shock waves. Waves in air or water are generally influenced most greatly by convection and so the convection equation represent their true nature rather well. When two waves are created at different spots also have different speeds and heights, it is possible for them to collide at a particular point in space and time. Our wave demands that $U$ equals its value of $U_0$ while the other demands that it be equal to some other value of $U_0$. The confusion creates a shock.

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