Math 2E03- Introduction to Modelling

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Problem : Assume that a colony of algae are being carried along a channel by a stream moving at a speed c. At a certain point along the channel, the colony of algae encounters, for a distance L, a souce of nutrients sufficiently plentiful to ensure exponential growth. Drive a partial differential equation (PDE) to model this situation. Detremine the solution of this differntial equation and deduce the relationship between the growth rate of the algae r, the length of nutrient interval L, and the speed of the stream c for which the colony population will double during the time it spends under nourishment.

Solution: This gives

$$\frac{\partial V}{\partial t} = rV - c\frac{\partial V}{\partial x}$$

Now to find out the solution we substitute $V = e^{rt}\tilde{V}$ in to given PDE, which leads to convection equation

$$\frac{\partial \tilde{V}}{\partial t} + c\frac{\partial \tilde{V}}{\partial x} = 0$$

So our solution will be of the form $V = e^{rt}\tilde{V}_0(x - ct)$.
On the second part of the problem -at what speed must be stream be flowing for the population during its time in the nutrient interval?. While the algae is in the nutrients it is growing expontially and since we are not leaving the nutrient area in the time interval in the question, we can ignore the convection term and simply use our exponential growth ODE

$$\frac{\partial V}{\partial t} = rV$$

The Solution is $V = e^{rt}V_0$. The time spent in the interval will be $\frac{L}{c}$. We substitute this into our equation and solve for c.

$$c = \frac{rL}{\log 2}$$

Diffusion

(a) Drunkard's Walk Let's say that a drunkard is stumbling home late one night, lurching right or left randomly. With each lurch he moves a distance r staright ahead and a distance s to the right or left. If we let straight ahead be in same direction as the x axis of graph, with y representing the left direction and -y representing the right, we can plot (given in class) it like that.

If we assume that each lurch (right or left) is independent of previous one (since he is stumbling randomly) and let S_n denote the displacement from his intended path at the n^{th} step, then we can see that his total lurch displacement after n steps is

$$y_n = S_1 + S_2 + \cdots + S_n$$

But what if we square y_n?

$$y_n^2 = (S_1 + S_2 + S_3 + \cdots + S_n)^2$$

$$y_n^2 = S_1^2 + S_2^2 + \cdots + S_n^2 + 2(S_1 S_2 + S_1 S_3 + \cdots)$$

The value of S_k is either s or -s. Thus all the terms in the parenthesis have an equal chance of being positive or being negative. They will all be the same magnitude of s^2, so they should all cancel out. The remaining terms will be

$$y_n^2 = ns^2$$

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Drunkard displacement in x direction after n steps should be rn.

$$x = rn$$

$y_n = \pm k\sqrt{(}x)$ where $k^2 = \dfrac{s^2}{r}$