Math 2E03- Introduction to Modelling

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If we were to specify a certain distribution of algae at $t=0$ is $\tilde{U}(x,0)=f(x)$, then we would also have an initial condition. Let's us impose our first boundary values

$$\begin{aligned}
\tilde{U}(0,t) = X(0)T(t) &= 0 \\
\implies X(0) &= 0 \\
B\sin(0) + C\cos(0) &= 0 \\
B(0) + C(1) &= 0 \\
C &= 0 \\
\implies X &= B\sin(\lambda x)
\end{aligned}$$

Note we can not let $T(t)=0$, because this would have to told true for all t and thn we would have the trivial case $\tilde{U}=0$ which represent the situation where there is no algae at all. We now impose the other boundary value:

$$\tilde{U}(L,t) = X(L)T(t) = 0$$

$$\implies X(L) = 0$$

$$B\sin(\lambda L) = 0$$

$$\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

$$\implies X_n = B_n \sin(\frac{n\pi}{L}x)$$

$$\implies T_n = A_n e^{-(\frac{n\pi}{L})^2 Dt}$$

$$\implies \tilde{U}_n = C_n \sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 Dt}$$

We now seek a solution involving the linear combination of solutions

$$\tilde{U}(x,t) = \sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 Dt}$$

Now our actual population of algae is

$$U(x,t) = e^{rt}\sum_{n=1}^{\infty} C_n \sin(\frac{n\pi}{L}x)e^{-(\frac{n\pi}{L})^2 Dt}$$

We can now answer the question posed in the problem. Note that the sin term simply oscillates- we can ignore it. It is the e term which will govern the overall growth of the population-if it increases so will the pop. but if it goes to zero the pop. will die out.

Let

$$r - (\frac{n\pi}{L})^2 D > 0$$

$$r > (\frac{n\pi}{L})^2 D$$

$$L > n\pi\sqrt{\frac{D}{r}}$$

For the pop. to grow faster than it diffuse out of the nutrient interval, the lenght of the interval must be greater than $n\pi\sqrt{\frac{D}{r}}$.

Now question arises why we took only constant $k = -\lambda^2$

Now we consider another two cases

case 1: $k = 0$ In this case $X(x) = Bx + C$ where B and C are some constants. Using boundary consitions $X(0) = X(L) = 0$ we get B=0=C which implies $\tilde{U}$ is identically 0. Which is not of our interest.

Case 2: $k = \lambda^2$

$$X(x) = Ae^{-\lambda x} + Be^{\lambda x}$$

Again using boundary consitions $X(0) = X(L) = 0$ we get B=0=C which implies $\tilde{U}$ is identically 0. Which is not of our interest.