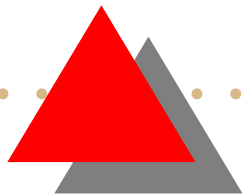





*Math 2E03- Introduction to
Modelling*

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If we were to specify a certain distribution of algae at $t = 0$ is $\tilde{U}(x, 0) = f(x)$, then we would also have an initial condition. Let's us impose our first boundary values

$$\tilde{U}(0, t) = X(0)T(t) = 0$$

$$\implies X(0) = 0$$


$$B \sin(0) + C \cos(0) = 0$$

$$B(0) + C(1) = 0$$

$$C = 0$$

$$\implies X = B \sin(\lambda x)$$

Note we can not let $T(t) = 0$, because this would have to be true for all t and then we would have the trivial case $\tilde{U} = 0$ which represents the situation where there is no algae at all. We now impose the other boundary value:


$$\tilde{U}(L, t) = X(L)T(t) = 0$$

$$\implies X(L) = 0$$

$$B \sin(\lambda L) = 0$$

$$\lambda L = n\pi$$


$$\lambda = \frac{n\pi}{L}$$

$$\implies X_n = B_n \sin\left(\frac{n\pi}{L}x\right)$$

$$\implies T_n = A_n e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

$$\implies \tilde{U}_n = C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$

We now seek a solution involving the linear combination of solutions


$$\tilde{U}(x, t) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$



Now our actual population of algae is

$$U(x, t) = e^{rt} \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi}{L}x\right) e^{-\left(\frac{n\pi}{L}\right)^2 Dt}$$


We can now answer the question posed in the problem. Note that the sin term simply oscillates- we can ignore it. It is the e term which will govern the overall growth of the population-if it increases so will the pop. but if it goes to zero the pop. will die out.

Let

$$r - \left(\frac{n\pi}{L}\right)^2 D > 0$$

$$r > \left(\frac{n\pi}{L}\right)^2 D$$

$$L > n\pi \sqrt{\frac{D}{r}}$$



For the pop. to grow faster than it diffuse out of the nutrient interval, the length of the interval must be greater than $n\pi\sqrt{\frac{D}{r}}$.

Now question arises why we took only constant $k = -\lambda^2$

Now we consider another two cases

case 1: $k = 0$ In this case $X(x) = Bx + C$ where B and C are some constants. Using boundary conditions $X(0) = X(L) = 0$ we get $B=0=C$ which implies \tilde{U} is identically 0. Which is not of our interest.

Case 2: $k = \lambda^2$

$$X(x) = Ae^{-\lambda x} + Be^{\lambda x}$$

Again using boundary conditions $X(0) = X(L) = 0$ we get $B=0=C$ which implies \tilde{U} is identically 0. Which is not of our interest.

