Math 2E03- Introduction to Modelling

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The Wave Equation

Consider, if you will, a violin string of length L and linear density (mass per length) σ, under a tension T. If the string runs along the x axis, we may define the displacemnet of the string as it vibrates as $U(x, t)$ as shown in the graph (given in class). If Δx is small enough, we may assume that the angles are quite small and that the horizontal componnets of the forces cancel each other out. This leaves the vertical forces $T \sin \theta_2 - T \sin \theta_1$. These forces must equal to given ma, where the mass is given by the linear density times the length and the acceleration may be written in differential form:

$$T \sin \theta_2 - T \sin \theta_1 \approx \sigma \Delta x \frac{\partial^2 U}{\partial t^2}$$

For very small angles, $\sin x \approx x \approx \tan x$ and in this case we will use the tangents since they may always be written as the difference of hight ∂U divided by the difference in width ∂x:

$$T \tan\theta_2 - T\tan\theta_1 \approx \sigma \Delta x \frac{\partial^2 U}{\partial t^2}$$

$$\frac{\frac{\partial U(x+\Delta x)}{\partial x} - \frac{\partial U(x)}{\partial x}}{\Delta x} \approx \frac{\sigma}{T}\frac{\partial^2 U}{\partial t^2}$$

$$\frac{\partial^2 U}{\partial t^2} = c^2 \frac{\partial^2 U}{\partial x^2} \text{ where } c = \sqrt{\frac{T}{\sigma}}$$

Solving Wave equation using Fourier's method

Again here we are looking for a solution of the form $\tilde{U}(x,t) = X(x)T(t)$

$$\text{In short form } U = XT$$

$$U_t = X\dot{T}$$

$$U_{tt} = X\ddot{T}$$

$$U_x = \dot{X}T$$

$$U_{xx} = \ddot{X}T$$

Substituting these values of U_{tt} and U_{xx} in wave equation we get

$$\Longrightarrow X\ddot{T} = c^2\ddot{X}T$$

$$\frac{\ddot{T}}{T} = c^2\frac{\ddot{X}}{X}$$

We again have two indpendent ODEs. Now we will set up these ODEs equal to the constant $-\lambda^2$.

$$\ddot{T} + c^2\lambda^2 T = 0$$

$$\ddot{X} + \lambda^2 X = 0$$

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So the solution of these two ODEs is

$$T = A\sin(\lambda ct) + B\cos(\lambda ct)$$

$$X = C\sin(\lambda x) + D\cos(\lambda x)$$

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In our example of violin string, our ends are fixed at all times and we will call the initial shape of our string $f(x)$. The another boundary condition is at $t = 0$ velocity at all points will be zero. –Now we will add boundary values $U(0,t) = U(L,t) = 0$. Let's us impose our first boundary value

$$U(0,t) = X(0)T(t) = 0$$

$$\implies X(0) = 0$$

$$B\sin(0) + C\cos(0) = 0$$

$$B(0) + C(1) = 0$$

$$C = 0$$

$$\implies X = B\sin(\lambda x)$$

We now impose the other boundary value:

$$U(L,t) = X(L)T(t) = 0$$

$$\implies X(L) = 0$$

$$C\sin(\lambda L) = 0$$

$$\lambda L = n\pi$$

$$\lambda = \frac{n\pi}{L}$$

$$\implies X_n = C_n \sin(\frac{n\pi}{L} x)$$

$$\implies T_n = A_n \sin(\lambda ct) + B_n \cos(\lambda ct)$$

$$\implies U_n = \left[A_n \sin(\frac{n\pi}{L}ct) + B_n \cos(\frac{n\pi}{L}ct)\right] \sin \frac{n\pi}{L}x$$