Math 2E03- Introduction to Modelling

Instrutor– Dr. Mani Mehra

Department of Mathematics and Statistics
McMaster Univ.

Density dependent population growth

Our earlier models and common sense shows that exponential growth can not continue forever. Here we return to the fundamental question of the causes and consequences of regulation of pop. growth. Hypothesis for pop. regulation :: Many hypothesis have been proposed for the causes of regulation of populations

- Pop. are limited by weather.

Density dependent population growth

Our earlier models and common sense shows that exponential growth can not continue forever. Here we return to the fundamental question of the causes and consequences of regulation of pop. growth.

Hypothesis for pop. regulation :: Many hypothesis have been proposed for the causes of regulation of populations

- Pop. are limited by weather.
- Pop. are limited by food supply.

Density dependent population growth

Our earlier models and common sense shows that exponential growth can not continue forever. Here we return to the fundamental question of the causes and consequences of regulation of pop. growth.

Hypothesis for pop. regulation :: Many hypothesis have been proposed for the causes of regulation of populations

- Pop. are limited by weather.
- Pop. are limited by food supply.
- Pop. are regulated by predators.

Density dependent population growth

Our earlier models and common sense shows that exponential growth can not continue forever. Here we return to the fundamental question of the causes and consequences of regulation of pop. growth. Hypothesis for pop. regulation :: Many hypothesis have been proposed for the causes of regulation of populations

- Pop. are limited by weather.
- Pop. are limited by food supply.
- Pop. are regulated by predators.
- Pop. are regulated by diseases.

Density dependent population growth

Our earlier models and common sense shows that exponential growth can not continue forever. Here we return to the fundamental question of the causes and consequences of regulation of pop. growth.

Hypothesis for pop. regulation :: Many hypothesis have been proposed for the causes of regulation of populations

- Pop. are limited by weather.
- Pop. are limited by food supply.
- Pop. are regulated by predators.
- Pop. are regulated by diseases.

Logistic model

The basic model we will examine takes the form

$$\frac{dN}{dt} = Nf(N) \tag{1}$$

Logistic model

The basic model we will examine takes the form

$$\frac{dN}{dt} = Nf(N) \tag{4}$$

where $f(N)$ is the per capita growth rate.

Logistic model

The basic model we will examine takes the form

$$\frac{dN}{dt} = Nf(N) \tag{7}$$

where $f(N)$ is the per capita growth rate. We now change the form of function $f(N)$ to include the effects of density dependence. We denote by K the value of pop. density at which the growth rate is 0. This is known as carrying capacity.

Logistic model

The basic model we will examine takes the form

$$\frac{dN}{dt} = Nf(N) \tag{10}$$

where $f(N)$ is the per capita growth rate. We now change the form of function $f(N)$ to include the effects of density dependence. We denote by K the value of pop. density at which the growth rate is 0. This is known as carrying capacity.

$$f(N) = r(1 - N/K) \tag{11}$$

This equation shows that the growth rate $f(N)$ is dependent on the population density N. Substituting the value of (2) in (1) we get

$$\frac{dN}{dt} = rN(1 - N/k) \tag{12}$$

Logistic model

Solving this equation for $N(t)$, we find

$$N(T) = \frac{N(0)e^{rT}}{1 + N(0)(e^{rT} - 1)/K} \qquad (13)$$

Some organism shows logistic growth during the first ten days of the experiment. However, the population then declines and appears to approach a second equilibrium phase. So how to modify logistic model to make it more realistic.

One way is to find the best fit is to calculate the parameters values that minimize the sum of the squares of the deviation of the model from the experimental data points. We can also ask how good the quantitative fit of the model to data. From some examples, we are led to the conclusion that we should not try to make too much of the quantitative aspects of the logistic model.

Qualitative Analysis

- Determine the values of the pop. density, $\hat{N}$, which are equilibria. Set

(14) $$\frac{dN}{dt} = 0$$

from this we get

$$r\hat{N}(1 - \hat{N}/k) = 0$$

which has the solution

$$\hat{N} = 0 \text{ and } \hat{N} = K$$

Qualitative Analysis

- Determine the values of the pop. density, $\hat{N}$, which are equilibria. Set

$$\frac{dN}{dt} = 0 \tag{16}$$

from this we get

$$r\hat{N}(1 - \hat{N}/k) = 0$$

which has the solution

$$\hat{N} = 0 \text{ and } \hat{N} = K$$

- See behavior near the equilibrium points.

$$\frac{dN}{dt} \approx rN \tag{17}$$

- Because $r > 0$, the solution of (15) is

$$N(t) = N(0)e^{rt}$$

We conclude that solutions grow exponentially when N is small.

Let n represent the deviation from the equilibrium, so

$$N = \hat{N} + n \tag{18}$$

Thus we rae intrested in finding how n changes with time. So

- Because $r > 0$, the solution of (15) is

$$N(t) = N(0)e^{rt}$$

We conclude that solutions grow exponentially when N is small.

Let n represent the deviation from the equilibrium, so

$$N = \hat{N} + n \tag{20}$$

Thus we rae intrested in finding how n changes with time. So

$$\frac{dN}{dt} = \frac{dn}{dt} = F(N) \text{ where } F(N) = rN(1 - N/K) \tag{21}$$

Now we need to approximate $F(N)$ near the equilibrium, $\hat{N}$.

- Because $r > 0$, the solution of (15) is

$$N(t) = N(0)e^{rt}$$

We conclude that solutions grow exponentially when N is small.

Let n represent the deviation from the equilibrium, so

$$N = \hat{N} + n \tag{22}$$

Thus we rae intrested in finding how n changes with time. So

$$\frac{dN}{dt} = \frac{dn}{dt} = F(N) \text{ where } F(N) = rN(1 - N/K) \tag{23}$$

Now we need to approximate $F(N)$ near the equilibrium, $\hat{N}$.

We use a Taylor series

$$F(\hat{N}+n) \approx F(\hat{N}) + n\frac{dF}{dN}|_{N=\hat{N}} \tag{24}$$

We use a Taylor series

(27) $$F(\hat{N}+n) \approx F(\hat{N}) + n\frac{dF}{dN}|_{N=\hat{N}}$$

Since $\hat{N}$ is an equilibrrium, $F(\hat{N}) = 0$. Thus we conclude

(28) $$\begin{aligned}\frac{dn}{dt} &\approx n\frac{dF}{dN}|_{\hat{N}} \\ &\approx n(r - 2rN/K)|_{\hat{N}}\end{aligned}$$

We use a Taylor series

(30) $$F(\hat{N}+n) \approx F(\hat{N}) + n\frac{dF}{dN}|_{N=\hat{N}}$$

Since $\hat{N}$ is an equilibrrium, $F(\hat{N}) = 0$. Thus we conclude

(31) $$\begin{aligned}\frac{dn}{dt} &\approx n\frac{dF}{dN}|_{\hat{N}} \\ &\approx n(r - 2rN/K)|_{\hat{N}}\end{aligned}$$

Near the equlibrium point $\hat{N} = 0$ we have seen earlier. Now near the another equilibrium point $\hat{N} = K$

(32) $$\frac{dn}{dt} \approx -rn$$

Thus n represents the deviation from the equilibrium $\hat{N} = K$,

$$n(t) = n(0)e^{-rt} \tag{33}$$

The basic conclusion from the qualitative analysis do not depend on the exact form of the logistic. And the equilibrium point $\hat{N} = 0$ is unstable, and that the equilibrium point $\hat{N} = K$ is stable.
Again behavior of some population whose number increses and decresses in a relatively regular fashion can not be explained by the logistic model.