(4) $u(x,t) = \sum A_n \cos \frac{n\pi t}{L} \sin \frac{n\pi x}{L}$

$f(x) = \sum A_n \sin \frac{m\pi x}{L}$

$$\int_0^L x \sin \frac{m\pi x}{L}\, dx = \sum A_n \int_0^L \sin \frac{n\pi x}{L} \sin \frac{m\pi x}{L}\, dx.$$

$n = m.$

$$A_n \int_0^L \sin^2 \frac{n\pi x}{L}\, dx = \int_0^L x \sin \frac{n\pi x}{L}\, dx$$

$$A_n \cdot \frac{L}{2} = \left[-x \frac{L}{n\pi} \cos \frac{n\pi x}{L} \right|_0^L + \left[\left(\frac{L}{n\pi}\right)^2 \sin \frac{n\pi x}{L} \right]_0^L$$

$$= -\frac{L^2}{n\pi} \cos n\pi$$

$$\Rightarrow \quad A_n = -\frac{2L}{n\pi} \cos n\pi$$

9

$$\frac{dV}{dt} = r_{in} - r_{out}$$
$$= 6 - 5$$

$\Rightarrow V = t + K$

at $t = 0$ $K = 1000$

$\Rightarrow V = t + 1000$

Now $\frac{dy}{dt} = r_{in} c_{in} - r_{out} \cdot \frac{y}{t+1000}$.

$$= 6 - \frac{5y}{t+1000}$$

$\frac{dy}{dt} + \frac{5y}{t+1000} = 6$

IF $= e^{\int \frac{5}{t+1000} dt}$

$= e^{5 \ln(t+1000)}$

$= (t+1000)^5$

$y \cdot (t+1000)^5 = \cancel{6} \cdot \frac{(t+1000)^6}{\cancel{6}} + C$

$y(t) = (t+1000) + c(t+1000)^{-5}$

using $y(0) = 0$.

$0 = 1000 + c(1000)^{-5}$

$c = -(1000)^6 \Rightarrow y(t) = (t+1000) - (1000)^6 (t+1000)^{-5}$

Concentration is

$$\frac{y(t)}{t+1000} = 1 - (1000)^6 (t+1000)^{-6} \text{ kg/L}$$

Problem 6.

$$N(t) = e^{rt} N(0)$$

suppose at $t = 0$ $N(t) = 3.93$ million

$$\Rightarrow e^{r \times 0} \cdot N(0) = 3.93$$

$$\Rightarrow N(0) = 3.93 \text{ million}$$

again at $t = 100$. $N(t) = 62.95$

$$\Rightarrow 62.95 = e^{100r} \cdot 3.93$$

$$\Rightarrow e^{100r} = \frac{62.95}{3.93}$$

$$= 16.0178$$

$$100r = 2.7737$$

$$r = .0277$$

$$\Rightarrow N(t) = 3.93 \cdot e^{.0277 \cdot t}$$

4. The n^{th} generation will have a population equal to the growth rate constant r times the previous pop. minus the number which will be harvested.

$$a_n = r a_{n-1} - p$$

Now question is can the pop. remain constant. For this is to be true.

$$a_n = a_{n-1} = a^*$$

where a^* is the value of constant pop.

Now $a^* = r a^* - p$

$$(r-1) a^* = p$$

$$a^* = \frac{p}{r-1}$$

$$\Rightarrow \quad a_n = a^* + \theta_n$$

$$(a^* + \theta_n) = r(a^* + \theta_{n-1}) - p$$

$$\theta_n = (r-1) a^* + r\theta_{n-1} - p$$

$$\theta_n = r\theta_{n-1} + \left(r-1 \cdot \frac{p}{r-1} - p\right)$$

$$\Rightarrow \quad \theta_n = r\theta_{n-1}$$

$$\Rightarrow \quad \theta_n = r^n \theta_0$$

$$\Rightarrow \quad a_n - a^* = r^n (a_0 - a^*)$$

$$a_n = r^n\left(a_0 - \frac{p}{r-1}\right) + \frac{p}{r-1}$$

(8) Problem 2

Let $X(t)$ denote the mass of carbon-14 present in burnt food of the campfire. Then carbon-14 decays at a rate proportional to its mass.

$$\frac{dX}{dt} = -\alpha X.$$

$$X(t) = X_0 e^{-\alpha t}$$

$$t_{1/2} = \frac{\log 2}{\alpha} \Rightarrow 5600 = \frac{\log 2}{\alpha}$$

$$\alpha = \frac{\log 2}{5600}$$

$$= 1.24 \times 10^{-4}$$

$$\Rightarrow X(t) = X_0 e^{-\alpha t}$$

$$.02 X_0 = X_0 e^{-.000124 t}$$

$$\Rightarrow e^{-.000124 t} = .02.$$

$$\Rightarrow t = 31{,}549 \text{ years}$$

(approximately)

Problem. 4

$$\frac{dT}{d\theta} = -k(T - Ta)$$

$$k = -\frac{1}{t}\ln\left(\frac{T - Ta}{T_0 - Ta}\right)$$

$T_0 = 70^\circ F$

$Ta = 32^\circ F$

$t = 15 \qquad T = 60$

$$\Rightarrow k = -\frac{1}{15}\ln\left(\frac{60-32}{70-32}\right)$$

$$= -\frac{1}{15}\ln\left(\frac{28}{38}\right)$$

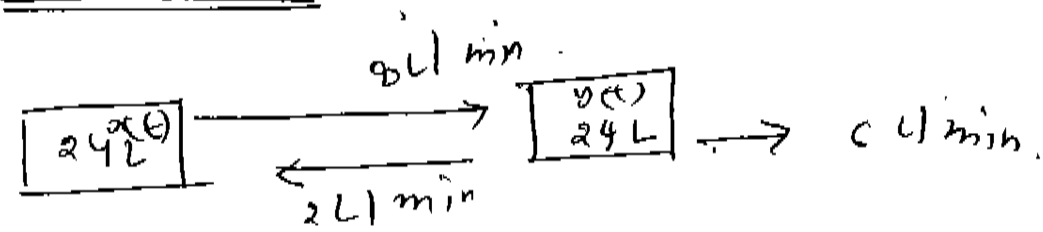
$$\simeq .0204$$

$\Rightarrow$ at $T = 56 \quad t = ?$

$$t = -\frac{1}{.0204}\ln\left(\frac{56-32}{38}\right)$$

$$\simeq 22.6 \text{ min}$$

Problem 10.



$$\frac{dx}{dt} = -\frac{8x}{V} + \frac{2y}{V} = -\frac{x}{3} + \frac{y}{12} \qquad V = 24.$$

$$\frac{dy}{dt} = \frac{8x}{V} - \frac{6y}{V} - \frac{2y}{V}$$

$$= \frac{x}{3} - \frac{y}{3}$$

In short form

$$12\dot{x} + 4x = y \quad \text{— (1)}$$

$$3\dot{y} - x + y = 0 \quad \text{— (2)}$$

eliminating y from eq. (2)

$$3(12\dot{x} + 4x)^{\cdot} - x + (12\dot{x} + 4x) = 0$$

$$\Rightarrow \quad 12\ddot{x} + 8\dot{x} + x = 0$$

$$\Rightarrow \quad 12\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + x = 0$$

solution is

$$x(t) = c_1 e^{-t/2} + c_2 e^{-t/6}.$$

$$y(t) = -2c_1 e^{-t/2} + 2c_2 e^{-t/6}.$$

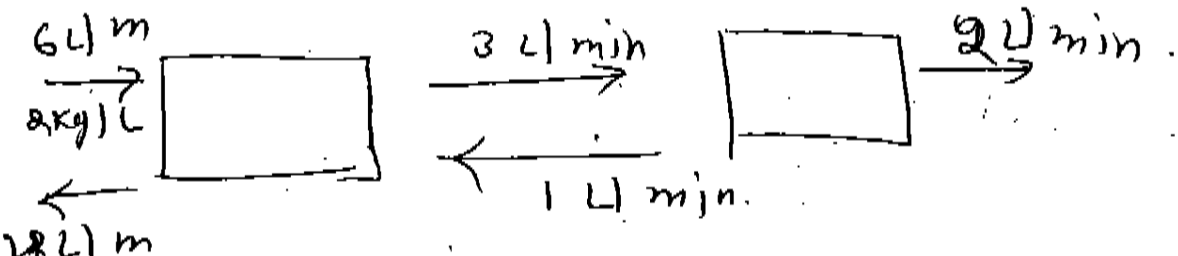
$$x_0 = c_1 + c_2$$
$$y_0 = -2c_1 + 2c_2$$

$$4c_2 = 2x_0 + y_0$$

$$\Rightarrow \quad c_2 = \frac{2x_0 + y_0}{4}$$

$$c_1 = x_0 - \left(\frac{2x_0 + y_0}{4}\right)$$
$$= \frac{2x_0 - y_0}{4}$$

Problem 11



$$\frac{dx}{dt} = 6\times 2 - \frac{3x}{V} + \frac{y}{V} - \frac{4x}{V} \qquad V = 100L$$

$$= 12 + \frac{y}{100} - \frac{7x}{100}$$

$$\frac{dy}{dt} = \frac{3x}{V} - \frac{2y}{V} - \frac{y}{V}$$

$$= \frac{3x}{100} - \frac{3y}{100}$$

Problem-12.

$$\frac{dy}{dt} = 50e^{-10t} - ky.$$

$$\frac{dy}{dt} + ky = 50e^{-10t}$$

$$I = e^{kt}$$

soln $$y \cdot e^{kt} = 50\int e^{-10t}e^{kt}\,dt + A$$

$$y e^{kt} = 50 \cdot \frac{e^{(k-10)t}}{k-10} + A$$

$$y = 50\frac{e^{-10t}}{k-10} + Ae^{-kt}.$$

$$k = 2. \qquad y(0) = 40$$

$$\Rightarrow \quad 40 = \frac{50}{k-10} + A$$

$$A = 40 + \frac{50}{+8}$$

$$= \frac{320 + 50}{8}$$

$$= \frac{\cancel{370}}{\cancel{8}}\ \frac{185}{4}$$

$$\Rightarrow \quad y(t) = -\frac{50}{8}e^{-10t} + \frac{185}{4}e^{-2t}.$$