Comparison Between Different Numerical Methods for Discretization of PDEs-A Short Review

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Abstract. We compare three well known methods for solving the PDEs such as Finite Difference Method (FDM), Spectral Method, Wavelet Galerkin Method (WGM). We test all these methods on Advection Equation and Klein-Gordon Equation. We plot comparison and error graphs using MATLAB.

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INTRODUCTION

Partial differential equations (PDEs) have several applications in several fields such as: physics, fluid dynamic and geophysics. However it is not always possible to get the solution of PDEs in closed form, therefore we need the solution of PDEs for many fields. Then the numerical methods come into the picture. There are several numerical methods (e.g. Finite Difference Method, Spectral Method, Finite Element Method, Finite Volume Method, Wavelet Galerkin Method (WGM)). In this report we are presenting some numerical methods to solve the partial differential equations (PDEs).

NUMERICAL METHODS FOR SOLVING PDES

Here we give brief introduction about some numerical methods to solve the PDEs

Finite Difference Method

Finite Difference Method (FDM) is most commonly used method to solve a Ordinary Differential Equations (ODEs) and PDEs in a bounded domain. The basic idea of finite difference methods is simple: derivatives in differential equations are written in terms of discrete quantities of dependent and independent variables, resulting in simultaneous algebraic equations with all unknowns prescribed at discrete nodal points for the entire domain. For example, the order of convergence in second order FDM is (N^{-2}) where N is number of nodal points. In brief about FDM , the different unknowns are defined by their values on discrete (finite) grid and differential operators are replaced by difference operators using neighboring points. See [1, 2] for details.

Spectral Method

Spectral Method is generally used when our function is periodic. It gives much better approximation of solution (which is periodic) than any other method. In this method we find the solution of PDEs in fourier space. Order of convergence in spectral method is $O(e^{-cN})$ where c is constant and N is number of nodal points. In this method we have to use Discrete Fourier Transform to project the equation in fourier space and Inverse Discrete Fourier Transform to project back to physical space. In brief about spectral method, Utilizing basis functions which are infinitely differentiable and non vanishing on the entire domain (global support). See [3, 4] for details.

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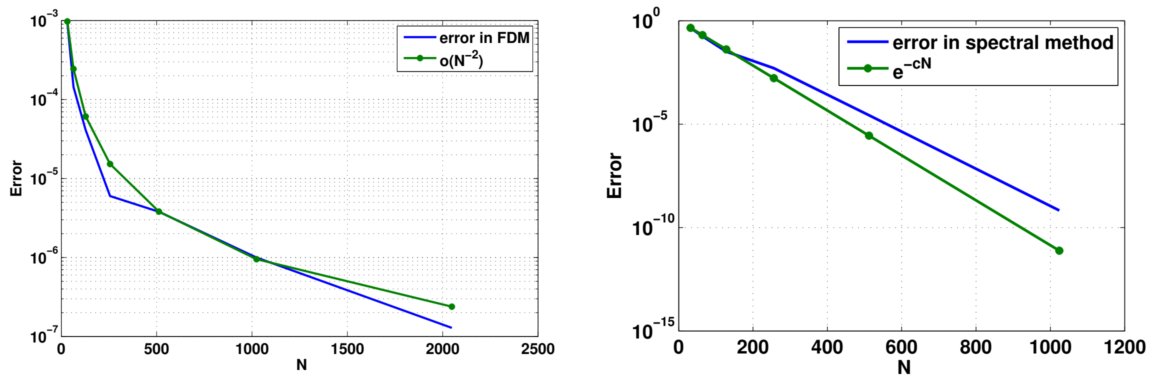

FIGURE 1. **Error analysis of Advection equation left: by FDM and right: by spectral method.**

Wavelet Galerkin Method

The Galerkin method defines an approximate solution to the weak form of the boundary value problems (BVPs) by restricting the problem to a finite-dimensional subspace. This has the effect of reducing the infinitely many equations to a finite system of equations. Notice that the equation has remained unchanged and only the space have changed. In the past two decades interest in wavelets has been nothing short of remarkable. Wavelets are used in many fields as: matrix compression and approximation theory. In the solution of differential equations, however wavelets have not, thus far, been able to replace other more traditional technique i.e. finite element methods. If we use wavelet basis [5] in place of basis function then this method becomes Wavelet Galerkin Method (WGM). See [6, 7, 8] for details.

NUMERICAL RESULTS AND DISCUSSION

Example 1 **(Advection Equation):** Solve the following PDE (Advection equation)

$$u_t = au_x$$

with initial condition $u(x,0) = \sin(2\pi x) + e^{-\alpha(x-1/2)^2}$, where $x \in [0,\ 1]$, $a = 0.01$ and $\alpha = 2\times 10^4$. We check the order of error in all methods. As we know that the order of error in FDM is N^{-2}, in Spectral method e^{-cN} and in WGM it is 2^{-jD}. Where N is number of nodal points, $c = 2\times 10^{-3}$ a constant and D is order of wavelets. Here we plot the error in l_2 norm with respect to N. Now observe that from left of Figure 1, at $N = 500$ FDM gives $\approx 10^{-5}$ accuracy which is near about to satisfy its order N^{-2}, and from right of Figure 1 spectral method gives $\approx 10^{-5}$ accuracy which satisfy its order e^{-cN}, while from Figure 2, WGM gives $\approx 10^{-8}$ accuracy, which satisfy its order 2^{-jD}, where all symbols have their usual meaning. Hence we can say that for Advection equation WGM gives better accuracy in comparison to other traditional methods (FDM, spectral method).

Example 2 **(The Klein-Gordon equation):** We consider the Klein-Gordon equation [9]

$$u_{tt} - u_{xx} + bu + g(u) = f(x,t)$$

subject to initial conditions

$$u(x,0) = h(x),\ u_t(x,0) = g(x),$$

where u is a function of x and t, b is real, g is a nonlinear, and h is known analytic function. The Klein-Gordon equation plays an important role in mathematical physics. The equation has attracted much attention in studying solitons and condensed matter physics in investigating the interaction of solitons in a collisionless plasma, the recurrence of initial states, and in examining the nonlinear wave equations.

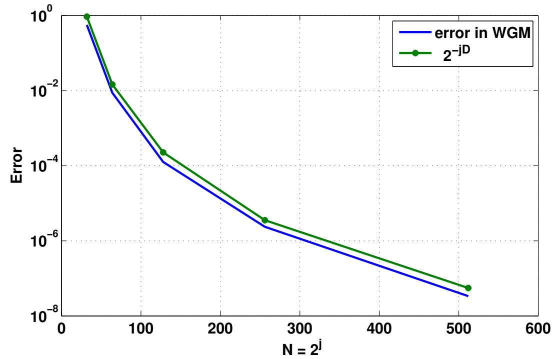

FIGURE 2. **Error analysis of Advection equation by WGM.**

We take b = -1 , $g(u)$ = 0 and $f(x,t)$ = 0 with $h(x) = 1 + \sin x$. Thus we have the following equation:

$$u_{tt} - u_{xx} = u,$$

subject to initial conditions $u(x,0) = 1 + \sin x$, $u_t(x,0) = 0$. The analytic solution of the following equation is $u(x,t) = \sin x + \cosh t$. Now we solve the Klien-Gordon equation by FDM, Spectral Method, and WGM and compare their solution. We see from Figures 3 and 4, which is comparison of all given three method at time t=0.1, 0.2, 0.5, 1 . When time is increasing the WGM method give more accurate result compare to other method.

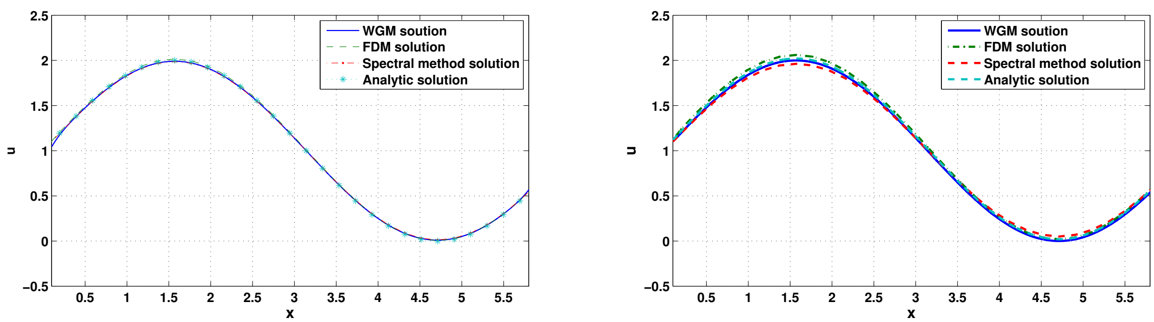

FIGURE 3. **All solution graph by FDM, spectral method, WGM and analytic solution of the Klein-Gordon equation left: at time t=0.1 and right: time t=0.2.**

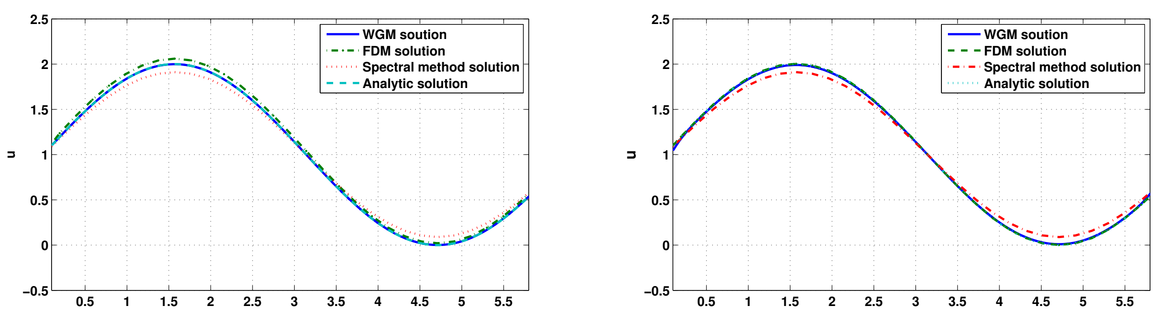

FIGURE 4. **All solution graph by FDM, spectral method, WGM and analytic solution of the Klein-Gordon equation left: at time t=0.5 and right: time t=1.**

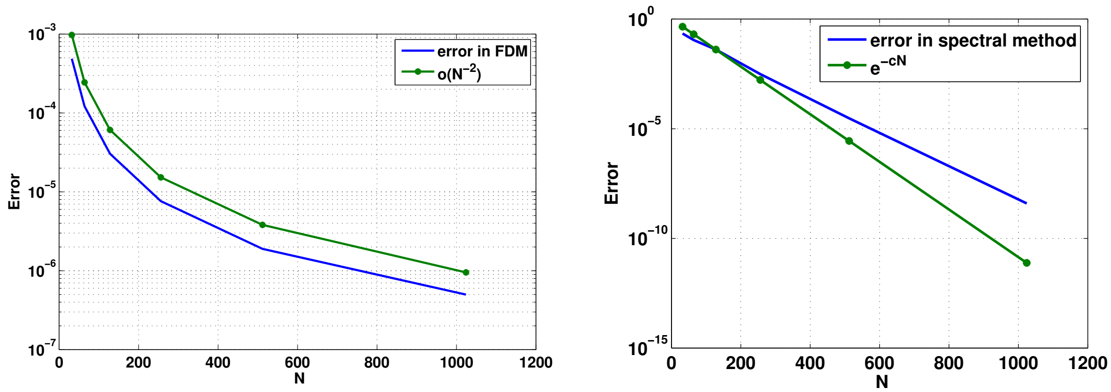

FIGURE 5. Error analysis of the Klein-Gordon equation left: by FDM and right: spectral method.

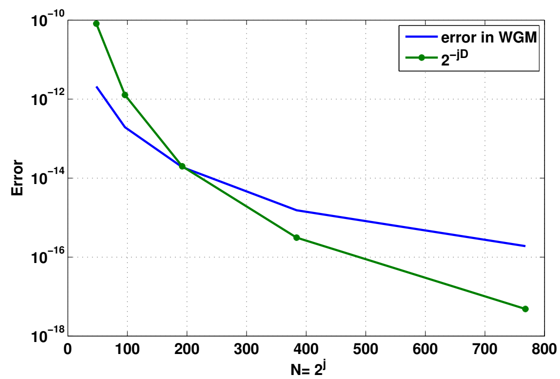

FIGURE 6. Error analysis of the Klein-Gordon equation by WGM.

Now observe that from Figure 5, at $N = 500$ FDM gives $\approx 10^{-5}$ accuracy, which is near about to satisfy its order N^{-2} and from right of Figure 5 spectral method gives $\approx 10^{-5}$ accuracy, which is near about to satisfy its order e^{-cN}, while from Figure 6, WGM gives $\approx 10^{-14}$ accuracy,satisfy its order 2^{-jD}. Hence we can say that for the Klein-Gordon equation WGM gives better accuracy in comparison to other traditional methods (FDM, spectral method).

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