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WAVELET-TAYLOR GALERKIN METHOD FOR THE BURGERS EQUATION*

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Abstract.

In this paper, we propose a wavelet-Taylor Galerkin method for the numerical solution of the Burgers equation. In deriving the computational scheme, Taylor-generalized Euler time discretization is performed prior to wavelet-based Galerkin spatial approximation. The linear system of equations obtained in the process are solved by approximate-factorization-based simple explicit schemes, and the resulting solution is compared with that from regular methods. To deal with transient advection-diffusion situations that evolve toward a convective steady state, a splitting-up strategy is known to be very effective. So the Burgers equation is also solved by a splitting-up method using a wavelet-Taylor Galerkin approach. Here, the advection and diffusion terms in the Burgers equation are separated, and the solution is computed in two phases by appropriate wavelet-Taylor Galerkin schemes. Asymptotic stability of all the proposed schemes is verified, and the L_∞ errors relative to the analytical solution together with the numerical solution are reported.

AMS subject classification (2000): 65M70.

Key words: Taylor–Galerkin method, wavelets, time marching scheme, splitting-up method, Burgers equation.

1 Introduction.

The nonlinear parabolic equation $\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial^2 x}$ known as the Burgers equation is one of the simplest combining both nonlinear propagation and diffusive effects. It represents a first step in the hierarchy of approximation of Navier–Stokes equations. Solutions to this equation exhibit a delicate balance between the nonlinear advection and the diffusion terms. This equation is often used to test numerical methods because an analytical expression for its solution is available for different sets of boundary and initial conditions, and these solutions are known to develop very sharp gradients that are difficult to reproduce with numerical methods.

Wavelet methods are a new numerical tool for solving partial differential equations (PDEs). Wavelets have many attractive features: orthogonality, arbitrary

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regularity, and good localization. Wavelet bases seem to combine the advantages of both spectral and finite element bases. Wavelet analysis became important due to their successful application in signal and image processing during the 1980s. The study of wavelets attained its present growth after the mathematical analysis of wavelets by Stromberg (1981) [22], Grossmann and Morlet (1984) [11], and Meyer (1985) [20]. The multiresolution analysis of Mallat [19] and Meyer [21] in (1989) lead to the Daubechies (1988) orthonormal family of wavelets. As wavelet theory progressed and more tools became available, their use spread to other areas besides signal processing.

A comparison of numerical solutions of the Burgers equation by the spectral method and finite difference method (FDM) can be found in [3]. Since the birth in the 1980s of wavelet methods, several works ([18, 14]) have been devoted to the numerical solution of the Burgers equation using these methods. Most of these works are based on the Galerkin projection to approximate the equation. In general, the treatment of the nonlinear term is done using a pseudospectral technique using some analytical interpolations. It results in a permanent backward and forward motion between the physical and the wavelet space. As in the case of spectral approximation, a method of collocation that allows working only in the physical space would be suitable. Vasilyev et al. [23] have presented an application of wavelet collocation to the Burgers equation.

In the conventional numerical approach to transient problems the accuracy gained in using high-order spatial discretization is partially lost due to use of low-order time discretization schemes. Here usually spatial approximation precedes temporal discretization. The reversed order of discretization can lead to better time-accurate schemes with improved stability properties. The fundamental concept behind the Taylor–Galerkin approach is to incorporate more analytical information into the numerical scheme in the most direct and natural way, so that the technique may be regarded as an extension to PDEs of the Obrechkoff methods [12] for ordinary differential equations. In fact this concept is not new, and similar procedures have already been considered in the context of finite difference methods [15, 16, 17] and also in conjunction with a spectral type of spatial representation [10]. Later Donea [6, 8] used it in deriving a time-accurate finite element scheme. Their approach consists primarily in extending the Taylor series in the time increment to the third order before spatial discretization. This procedure has not been implemented so far in the wavelet approach to PDEs. Wavelet bases are known for their spatial accuracy. The aim of this paper is to formulate a wavelet-Taylor Galerkin method (W-TGM) for the Burgers equation and generalized Burgers equation in one dimension. The approximate factorization technique [7], when applied to the linear system resulting from applying the numerical scheme, will lead to a simple explicit scheme for solving the linear system. We compute the solution by explicit scheme and compare it with the solution obtained from a usual matrix inversion. Next, a splitting-up method in conjunction with appropriate wavelet-Taylor Galerkin schemes for advection and diffusion phases for solving the Burgers equation is proposed. Usually, in dealing with transient situations that evolve toward a highly convective steady state, the W-TGM reduces to a wavelet Galerkin method as the temporal term vanishes.

Especially in these cases the splitting-up method will be effective. This study, we believe, can lead to the development of robust algorithms for the solution of nonlinear multidimensional systems such as the Navier–Stokes equations.

The outline of the paper is as follows. In Section 2 we summarize some basics of wavelet analysis. In Section 3 we consider the application of W-TGM to the Burgers equation and generalized Burgers equation. This section is also devoted to treating the approach to the steady state accurately. We propose the use of a wavelet-Taylor Galerkin splitting-up method. We also discuss approximate factorization techniques and explicit schemes in Section 4. In Section 5 we study numerically the stability of the algorithm in these cases. Numerical results are provided in Section 6. Finally, in Section 7 we draw a number of conclusions based on our results.

2 Wavelet preliminaries.

2.1 Compactly supported wavelets.

The class of compactly supported wavelet bases was introduced by Daubechies in 1988 [5]. They are an orthonormal basis for functions in $L^2(R)$. A "wavelet system" consists of the function $\phi(x)$ and the function $\psi(x)$ referred to as wavelet functions. We define translates of $\phi(x)$ as

$$\phi_i(x) = \phi(x - i)\,. \tag{2.1}$$

Multiresolution analysis (MRA) is the theory used by Daubechies to show that for any nonnegative integer n there exists an orthogonal wavelet with compact support such that all the derivatives up to order n exist. MRA describes a sequence of nested approximation spaces V_j in $L^2(R)$ such that closure of their union equals $L^2(R)$. MRA is characterized by the following axioms:

$$\begin{gathered}
\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(R) \\
\overline{\bigcup_{j=-\infty}^{j=\infty} V_j} = L^2(R) \\
\bigcap_{j\in Z} V_j = 0 \\
f \in V_j \text{ if and only if } f(2(.)) \in V_{j+1} \\
\phi(x-k)_{k\in Z} \text{ is an orthonormal basis for } V_0\,.
\end{gathered} \tag{2.2}$$

We define W_j as the orthogonal complement of V_j in V_{j+1}, i.e., $V_j \perp W_j$ and

$$V_{j+1} = V_j + W_j\,. \tag{2.3}$$

2.1.1 Construction of wavelets from multiresolution analysis.

The construction of the wavelet basis stands on the fact that during the process of refinement in the approximation one wants to only store the improvement from approximation j to approximation $j+1$. Mathematically, one introduces at each step j subspace W_j, defined as the orthogonal complement of V_j in V_{j+1}. Then one has the fundamental theorem proved by Mallat and Meyer:

THEOREM 2.1. *There exists a function of W_0 such that $\psi(x-k), k \in Z$ is an orthonormal basis of W_0. The function ψ has the regularity properties*

$$\int x^k \psi(x)dx = 0 \text{ for } 0 \le k \le D/2-1\,. \tag{2.4}$$

The function $\phi_{j,k}(x) = 2^{j/2}\phi(2^j x-k)_{k\in Z}$ is an orthonormal basis for V_j, and the function $\psi_{j,k} = 2^{j/2}\psi(2^j x-k)_{k\in Z}$ is an orthonormal basis for W_j. Each member of the wavelet family is determined by the set of constants a_k (low-pass filter) by the dilation equation

$$\phi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k \phi(2x-k) \tag{2.5}$$

and the equation

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k \phi(2x-k)\,, \tag{2.6}$$

where D is the order of the wavelet and $b_k = (-1)^k a_{D-1-k}, k = 0, 1, \cdots, D-1$. The scaling function ϕ and wavelet function ψ satisfy the following relations:

$$\int_{-\infty}^{\infty} \phi(x)dx = 1\,,$$

$$\int_{-\infty}^{\infty} \phi_{j,k}(x)\phi_{j,l}(x)dx = \delta_{k,l}\,,$$

$$\int_{-\infty}^{\infty} \psi_{i,k}(x)\psi_{j,l}(x)dx = \delta_{i,j}\delta_{k,l}\,,$$

$$\int_{-\infty}^{\infty} \phi_{i,k}(x)\psi_{j,l}(x)dx = 0 \quad j \ge i\,.$$

2.2 Periodized wavelets.

Let $\phi \in L^2(R)$ and $\psi \in L^2(R)$ be the scaling and wavelet function from a MRA as defined in Section 2.1. For any $j, l \in Z$ we define the 1-periodic scaling function

$$\tilde{\phi_{j,l}}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n) = 2^{j/2}\sum_{n=-\infty}^{\infty} \phi(2^j(x+n)-l), \quad x \in R \tag{2.7}$$

and the 1-periodic wavelet

$$\tilde{\psi_{j,l}}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n) = 2^{j/2} \sum_{n=-\infty}^{\infty} \psi(2^j(x+n)-l), \quad x \in R. \tag{2.8}$$

The 1 periodicity can be verified as follows:

$$\tilde{\phi_{j,l}}(x+1) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n+1) = \sum_{m=-\infty}^{\infty} \phi_{j,l}(x+m) = \tilde{\phi_{j,l}}(x)$$

and similarly $\tilde{\psi_{j,l}}(x+1) = \tilde{\psi_{j,l}}(x)$.

2.3 Connection coefficients.

Any numerical scheme for solving differential equations must adequately represent the derivatives and nonlinearities of the unknown function. In the case of wavelet basis, these approximations give rise to certain L_2-inner products of the basis functions, their derivatives, and their translates, called the connection coefficients. Specific algorithms have been devised by Latto et al. [13]. In the most general case we allow ϕ_l to be differentiated, which gives rise to the n-term connection coefficients:

$$\wedge(l_1, l_2, \cdots, l_n, d_1, d_2, \cdots, d_n) = \wedge^{d_1 d_2 \cdots d_n}_{l_1 l_2 \cdots l_n} = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \phi_{l_i}^{d_i}(x).$$

2.4 Projection onto space V_j.

Let $P_{V_j} f$ be the projection of a function f onto V_j and

$$P_{V_j} f(x) = \sum_{k=-\infty}^{\infty} c_{j,k}^{(d)} \tilde{\phi}_{j,k}(x), \quad x \in R. \tag{2.9}$$

2.4.1 Interpolation.

Using interpolation is also a popular choice for projecting f onto V_j, either as part of a collocation method, for instance in [9], but also within a Galerkin scheme [1], and it is the method implemented in this paper. The coefficients $c_{j,k}$ are chosen such that the projection of f onto V_j and f coincides at node points of level j:

$$f(l/2^j) = \sum_{k=0}^{2^j-1} c_{j,k} \tilde{\phi}_{j,k}(l/2^j), \quad l = 0, \cdots, 2^j - 1.$$

This can be rewritten as

$$f(l/2^j) = \sum_{k=0}^{2^j-1} c_{j,k} \tilde{\phi}_{j,k-l}(l/2^j), \quad l = 0, \cdots, 2^j - 1.$$

Therefore, calculating the coefficients $c_{j,k}$ reduces to solving a matrix equation

$$F_j = T_j c_j \,,$$

where F_j is the vector of components $f_{j,l} = f(l/2^j)$ for $l = 0, \cdots, 2^j - 1, c_j$ is the vector of coefficients $c_{j,k}$ for $k = 0, \cdots, 2^j - 1$, and T_j is the circulant matrix of size $N = 2^j$.

Cost: If simple Gaussian elimination is used to solve this system, then the cost of finding $c_{j,k}$ is heavy: $O(n^3)$ operations, where n is the order of the matrix. Significantly better performance can be achieved using sparse matrix routines. However, because this system is circulant, using FFT (Fast Fourier Transform), the solution can be found in $O(nlog_2 n)$ operations [1].

3 Variant of wavelet-Taylor Galerkin method (W-TGM).

3.1 Wavelet-Taylor Galerkin method.

3.1.1 Burgers equation.

The simplest equation that models physical situations where both nonlinear advection and diffusion effects are important is the Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial^2 x}, \quad 0 \le x \le 1 \,. \tag{3.1}$$

We consider periodic boundary conditions for (3.1) and initial condition $u_0(x)$.

Time discretization: We have used second-order W-TGM because too many terms are introduced in the third-order time derivative term, especially for non-linear problems. Let us leave the spatial variable x continuous and discretize (3.1) in time by the following forward Taylor series expansion:

$$(u_t)^n = \frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2}u^n_{tt} - O(\Delta t^2) \,, \tag{3.2}$$

which includes first-order and second-order time derivatives. While the former is provided directly by (3.1), the latter can be obtained by taking the time derivative of the governing PDEs:

$$u_t = -\frac{1}{2}\partial_x(u^2) + \nu\partial_x^2 u \,. \tag{3.3}$$

The time derivative of (3.3) is

$$u_{tt} = -\partial_x(uu_t) + \nu\partial_x^2 u_t \,, \tag{3.4}$$

and the substitution of (3.3) and (3.4) into the Taylor series expansion (3.2) gives

$$\left[1 - \frac{1}{2}\Delta t\big(-\partial_x u^n - u^n\partial_x + \nu\partial_x^2\big)\right](u^{n+1} - u^n)/\Delta t = -\frac{1}{2}\partial_x(u^n)^2 + \nu\partial_x^2 u^n \,. \tag{3.5}$$

Spatial discretization: To obtain a fully discrete equation we apply the wavelet Galerkin method to (3.5) with an approximation of the form

$$u_j(x,t) = \sum_{k=0}^{2^j-1} c_{j,k}(t)\tilde{\phi}_{j,k}(x)\,, \tag{3.6}$$

where $c_{j,k}$ is the unknown coefficient of scaling function expansion. The associated wavelet Galerkin equations are

$$\begin{aligned} \frac{c_{j,l}^{n+1} - c_{j,l}^{n}}{\Delta t}\left(1 + \frac{\Delta t}{2}\left(2^{3j/2}\sum_k m_{l,k} - \nu(2^j)^2\sum_k \wedge_{lk}^{02}\right)\right) \\ = -2^{3j/2}\sum_k\sum_m c_{j,k}^n c_{j,m}^n \wedge_{lkm}^{001} + \nu(2^j)^2 \sum_k c_{j,k}\wedge_{lk}^{02} \end{aligned} \tag{3.7}$$

for $l = 0, 1, \cdots, 2^j - 1$, and $m_{l,k}$ is given by the following expressions involving the three-level connection coefficient given in Section 2.3:

$$m_{l,k} = \sum_m c_{j,m}^n \left(\wedge_{lkm}^{001} + \wedge_{lkm}^{010}\right).$$

3.1.2 Generalized Burgers equation.

Here we consider the generalized Burgers equation

$$u_t + u^\beta u_x + \lambda u^\alpha = \nu u_{xx}, \quad 0 \le x \le 1, t \ge 0 \tag{3.8}$$

for constants α, β, $\nu \ge 0$, and real λ, together with an initial condition $u_0(x)$ and periodic boundary conditions. We set $\alpha = \beta = 1$.

Time discretization: The forward Taylor series expansion includes a second-order term,

$$(u_t)^n = \frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2}u_{tt}^n - O(\Delta t^2)\,, \tag{3.9}$$

which includes first- and second-order time derivatives. While the former is provided directly by (3.8), the latter can be obtained by taking the time derivative of the governing PDEs:

$$u_t = -\frac{1}{2}\partial_x(u^2) + \nu\partial_x^2 u - \lambda u\,. \tag{3.10}$$

The time derivative of (3.10) is

$$u_{tt} = -\partial_x(uu_t) + \nu\partial_x^2 u_t - \lambda u_t\,, \tag{3.11}$$

and the substitution of (3.11) and (3.10) into the Taylor series expansion (3.9) gives

$$\Big[(1+\lambda\Delta t/2) - \frac{1}{2}\Delta t\big(-\partial_x u^n - u^n\partial_x + \nu\partial_x^2\big)\Big](u^{n+1}-u^n)/\Delta t = -\frac{1}{2}\partial_x(u^n)^2 + \nu\partial_x^2 u^n - \lambda u\,. \tag{3.12}$$

The spatial discretization of this semidiscrete equation is achieved by the wavelet Galerkin method (WGM).

3.2 Wavelet-Taylor Galerkin splitting-up method.

Here the basic idea is to test the strategy of the splitting-up method together with W-TGM for the advection-diffusion equation as it can be directly extended and used in multidimensional situations, for instance in solving Navier–Stokes equations. Hence, we again consider the Burgers equation, which is usually considered a reasonably simple substitute for Navier–Stokes equations though it is strictly not an advection-diffusion equation evolving toward a highly convective steady state. The problem is decomposed into a nonlinear advection problem followed by a pure diffusion problem. We will call it a W-TGMS scheme. Consider:

$$u_t + Au = 0\,, \tag{3.13}$$

where $A = A_1 + A_2$, $A_1 = u\partial_x$, and $A_2 = -\nu\partial_x^2$, and represent problem (3.13) on each time interval $\Delta t = t^{n+1} - t^n$ by

$$u_{\alpha t} + A_\alpha u_\alpha = 0, \quad \alpha = 1, 2\,, \tag{3.14}$$

with

$$u_1^{n+1} = u^n, \quad u_2^n = u_1^{n+1}, \quad u^{n+1} = u_2^{n+1}\,, \tag{3.15}$$

where each phase is treated by W-TGM.

3.2.1 Advection phase.

The advection problem corresponds to (3.14) with $\alpha = 1$:

$$u_{1t} + A_1 u_1 = 0\,. \tag{3.16}$$

The temporal discretization is achieved by second-order W-TGM

$$\frac{\left(u_1^{n+1} - u_1^n\right)}{\Delta t} = u_{1t}^n + \frac{1}{2}\Delta t u_{1t}^n + O(\Delta t^2)\,, \tag{3.17}$$

where $u_1^n = u^n$. The spatial discretization of this semidiscrete equation is achieved by the WGM.

3.2.2 Diffusion phase.

According to (3.14) with $\alpha = 2$, the diffusion problem reads

$$u_{2t} + A_2 u_2 = 0\,, \tag{3.18}$$

and a second-order temporal discretization is obtained by writing

$$\left(1 + \frac{1}{2}\Delta t A_2\right)\left(u_2^{n+1} - u_2^n\right)/\Delta t = -A_2 u_2^n\,, \tag{3.19}$$

where $u_2^n = u_1^{n+1}$ and $u_2^{n+1} = u^{n+1}$. A spatially discrete form of (3.19) is again obtained by the WGM. Periodic boundary conditions are incorporated into both the advection and diffusion phase numerical schemes.

4 Approximate factorization techniques and explicit schemes.

The W-TGM and W-TGMS require solving at each time step a system of linear algebraic equations of the form

$$AV = F\,, \tag{4.1}$$

where $V = u^{n+1} - u^n$ is the vector of nodal unknowns. $F = F(u^n)$ is a known vector and A varies with time and must be recomputed; the exact factorization of the matrix will lead to a significant computational expense. Therefore, approximate factorization techniques appear to be quite attractive for these applications.

In the present context it is indeed essential to retain the consistent character of the wavelet-Taylor Galerkin "mass" matrix A in the approximate factorization procedure. Let us consider the identity

$$A = L + (A - L)\,, \tag{4.2}$$

where L is the diagonal and positive matrix obtained from A by the row-sum technique.

$$L_{ii} = \sum_i A_{ii}, \quad L_{ij} = 0, j \neq i \tag{4.3}$$

Since L is diagonal and has positive entries, therefore

$$A = L^{\frac{1}{2}}(I + X)L^{\frac{1}{2}}\,, \tag{4.4}$$

where

$$X = L^{-\frac{1}{2}}(A - L)L^{-\frac{1}{2}}\,. \tag{4.5}$$

Then, under the assumption $\|X\| \leq 1$, the inverse of A in the form of (4.4) can be expressed by the following series:

$$A^{-1} = L^{-\frac{1}{2}}(I - X + X^2 - X^3 + \cdots)L^{-\frac{1}{2}}\,. \tag{4.6}$$

Truncating the series after X gives the following two-term approximation of the inverse of A:

$$A^{-1}(2) = 2L^{-1}\left(I - \frac{1}{2}AL^{-1}\right), \tag{4.7}$$

while retaining X^2 produces the three-term approximation

$$A^{-1}(3) = 3L^{-1}\left(I - AL^{-1} + \frac{1}{3}AL^{-1}AL^{-1}\right), \tag{4.8}$$

and so on. The successive approximations in the above approximate factorization technique can be generated by the following multipass algorithm. Consider the sequence of approximate solutions $V^{(g)}, g = 0, 1, \cdots, G$, defined as follows. Start from $V^{(0)} = 0$. Then, for $g = 0, 1, \cdots, G-1$, determine $V^{(g+1)}$ from $V^{(g)}$ by means of the "diagonal" linear system

$$LV^{(g+1)} = F - (A - L)V^{(g)}. \tag{4.9}$$

Finally, assume $V = V^{(G)}$. The approximation of (4.7) and (4.8) can be obtained by this simple procedure with $G = 2$ and $G = 3$, respectively. For $G = n$ we will call it an n-pass explicit scheme.

5 Theoretical stability of linearized schemes.

We use the notion of asymptotic stability of a numerical method as it is defined in [4] for a discrete problem of the form

$$\frac{dU}{dt} = LU,$$

where L is assumed to be a diagonalizable matrix.

Definition: The region of absolute stability of a numerical method is defined for the scalar model problem

$$\frac{dU}{dt} = \lambda U$$

to be the set of all $\lambda\Delta t$ such that $\|U^n\|$ is bounded as $t \to \infty$. Finally, we say that a numerical method is asymptotically stable for a particular problem if, for sufficiently small $\Delta t > 0$, the product of Δt times every eigenvalue of L lies within the region of absolute stability.

Forward (Euler) scheme: The region of absolute stability for this scheme is the circle of radius 1 and center $(-1,0)$.

6 Numerical results and discussions.

In this section we present the results of numerical experiments in which we compute the approximation to the solutions of the Burgers equation and the generalized Burgers equation.

6.1 Burgers equation.

The analytical solution to (3.1) subject to initial condition $u_0(x)$ after using a Cole–Hopf transformation is given by

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\xi}{t} \exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] \exp\left[-(2\nu)^{-1}\int_0^{\xi} u_o(\eta)d\eta\right] d\xi}{\int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\nu t}\right] \exp\left[-(2\nu)^{-1}\int_0^{\xi} u_0(\eta)d\eta\right] d\xi}. \tag{6.1}$$

We have integrated numerically (6.1) using the Gauss–Hermite quadrature in order to measure the accuracy of our method.

6.1.1 Case 1.

First, periodic boundary conditions are imposed together with a full sine wave initial condition ($u_0(x) = sin(2\pi x), x \in (0,1)$). The resultant solution is a stationary wave with the buildup of a steep front at the midpoint of the domain. Figure 6.1 shows the exact solution at various times for the value of $\nu = 10^{-2}/\pi$.

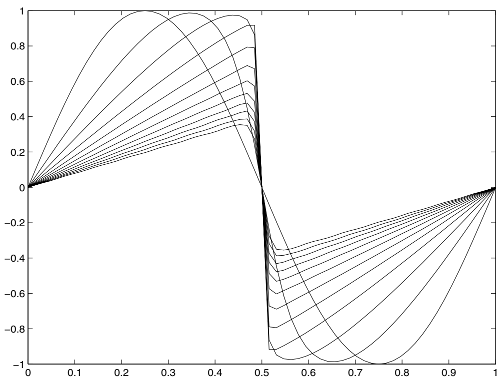


Figure 6.1: Exact solution of the Burgers equation plotted at each .1 time interval, starting from time $t = 0$ to 1.2.

Figure 6.2 shows the solution for the scales $4 \leq j \leq 9$, with a value $\nu = 10^{-2}/\pi$ of the viscosity. As time evolves, a shock develops, and the solution presents Gibbs-like oscillations for coarse scales. However, the oscillations are confined in the neighborhood of the shock. Apart from the shock area, the solution is correctly represented. This ability of wavelets to confine oscillations only to the vicinity of the shock has already been observed by [18, 2] in the case of the Galerkin method. It is a consequence of the good localization of the basis functions. In the Fourier method, for instance, where the basis functions have noncompact support, it is well known that oscillations spread in the whole do-

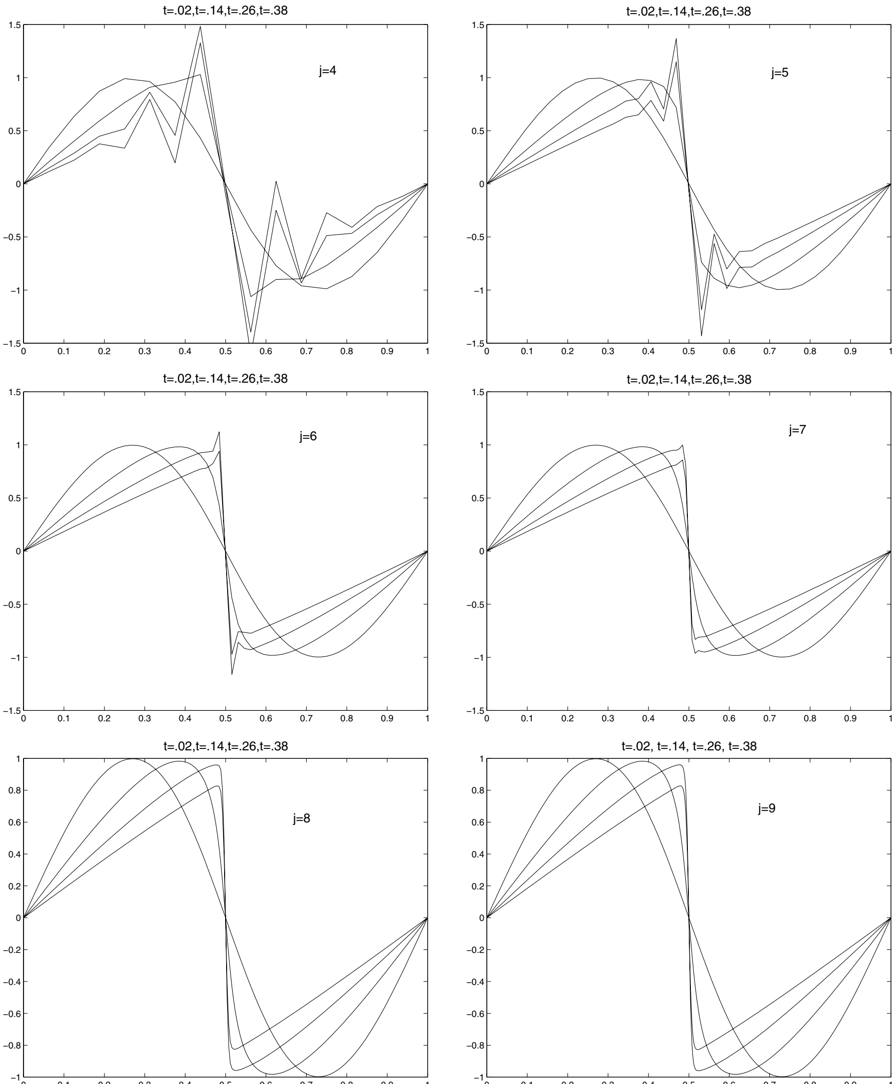


Figure 6.2: Numerical solution of the Burgers equation at various times and scales (j). The time step is $\Delta t = 10^{-4}$.

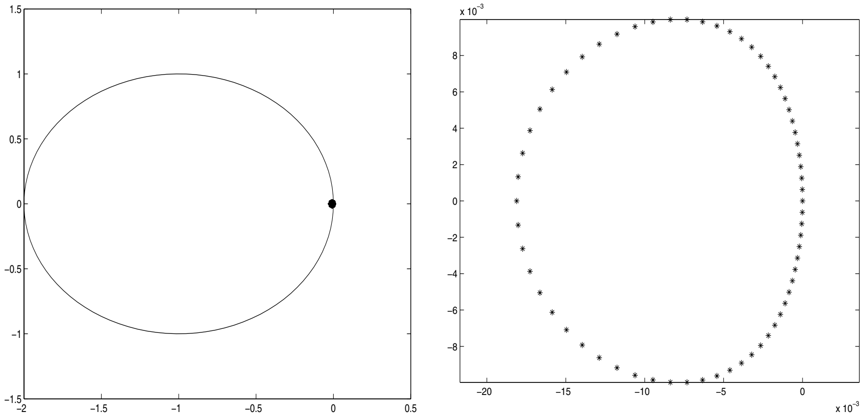
main unless a sufficient number of points are considered to resolve correctly the sharp gradient in the layer of the shock. The oscillations are due to the fact that the number of points in the shock layer whose thickness is very small is not sufficient to resolve correctly the large gradient that occurs here. In order to better represent the solution in the area of shocks, one needs to add more points here. One way to do this is to decrease the resolution. With smaller and smaller scales, one progressively resolves the shock, as can be seen in Figure 6.2. With scale $j = 7$ (i.e., 128 points), for instance, the numerical solution is smooth almost everywhere except at points immediately before and after the shocks.

All the results we present are obtained using a Daubechies $D6$ scaling function. Here WGM, W-TGM (W-TGM means implicit W-TGM), 3-pass explicit W-TGM, and W-TGMS were compared. The relative L_∞-errors are reported in Table 6.1. The explicit W-TGM scheme is found to reproduce almost exactly the results of the implicit scheme till time $t = 0.14$, except in the case $j = 7$. The difference in accuracy between W-TGM (implicit) and W-TGM (explicit) is more visible after time $t = 0.14$, especially for the case $j = 7$. The improvement of W-TGM methods with respect to the Euler–Galerkin method is apparent for time $t >= 0.26$. The reason for not gaining substantial improvement is that the part of the Burgers equation that plays the dominant role in this example is the steady-state one, i.e., the right-hand side of (3.3), and it is not modified by W-TGM. W-TGM (explicit) is seen to be superior to other methods with increasing t and j.

Table 6.1

j	Δt	at time:	WGM	W-TGM	W-TGM(explicit)	W-TGMS
4	10^{-4}	$t = 0.02$	0.0111	0.0111	0.0111	0.0111
		$t = 0.14$	0.1588	0.1588	0.1589	0.1587
		$t = 0.26$	0.6864	0.6856	0.6854	0.6854
		$t = 0.38$	0.6961	0.6955	0.6955	0.6952
5	10^{-4}	$t = 0.02$	0.0045	0.0045	0.0046	0.0045
		$t = 0.14$	0.0520	0.0523	0.0539	0.0522
		$t = 0.26$	0.4745	0.4737	0.4733	0.4729
		$t = 0.38$	0.3675	0.3676	0.3661	0.3669
6	10^{-4}	$t = 0.02$	0.0016	0.0016	0.0017	0.0016
		$t = 0.14$	0.0117	0.0117	0.0177	0.0117
		$t = 0.26$	0.2134	0.2127	0.2121	0.2108
		$t = 0.38$	0.1550	0.1550	0.1500	0.1535
7	10^{-4}	$t = 0.02$	0.0005	0.0005	0.0018	0.0005
		$t = 0.14$	0.0030	0.0030	0.0303	0.0030
		$t = 0.26$	0.0508	0.0503	0.0470	0.0502
		$t = 0.38$	0.0422	0.0422	0.0294	0.0419

For stability analysis we will consider the linearized Burgers equation $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial x^2} - \alpha\frac{\partial u}{\partial x}$, where the linearization coefficient α stands for the value of u. Because at the initial time $u_0(x) = sin(2\pi x)$ and because the amplitude of u decreases with time, we assume throughout that $|\alpha| \leq 1$. The region of absolute stability for this scheme is plotted in Figure 6.3.



(a) Absolute stability region for forward Euler and Δt times eigenvalues of L_6.

(b) Δt times eigenvalues of L_6.

Figure 6.3: Absolute stability region of the Burgers equation for case 1 where $\Delta t = 10^{-4}$, $\alpha = 1$, and $\nu = 10^{-2}/\pi$.

6.1.2 Case 2.

In this example we compute the solution of the Burgers equation using the initial condition $u_0(x) = sin(2\pi x) + \frac{1}{2}sin(4\pi x)$ and a periodic boundary condition, which leads to the formation of left and right moving shocks. Figure 6.4 shows the exact solution and Figure 6.5 shows the numerical solution for $4 \leq j \leq 7$. Here the difference in accuracy between the W-TGM and W-TGM explicit is more visible at time $t = 0.38$. The relative L_∞ errors are reported in Table 6.2. Here also we assume throughout that $|\alpha| \leq 1$. The region of absolute stability for this scheme is plotted in Figure 6.6.

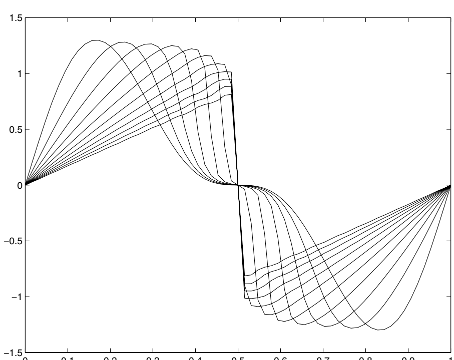


Figure 6.4: Exact solution of the Burgers equation plotted at each 0.05 time interval, starting from time $t = 0$ to 0.5.

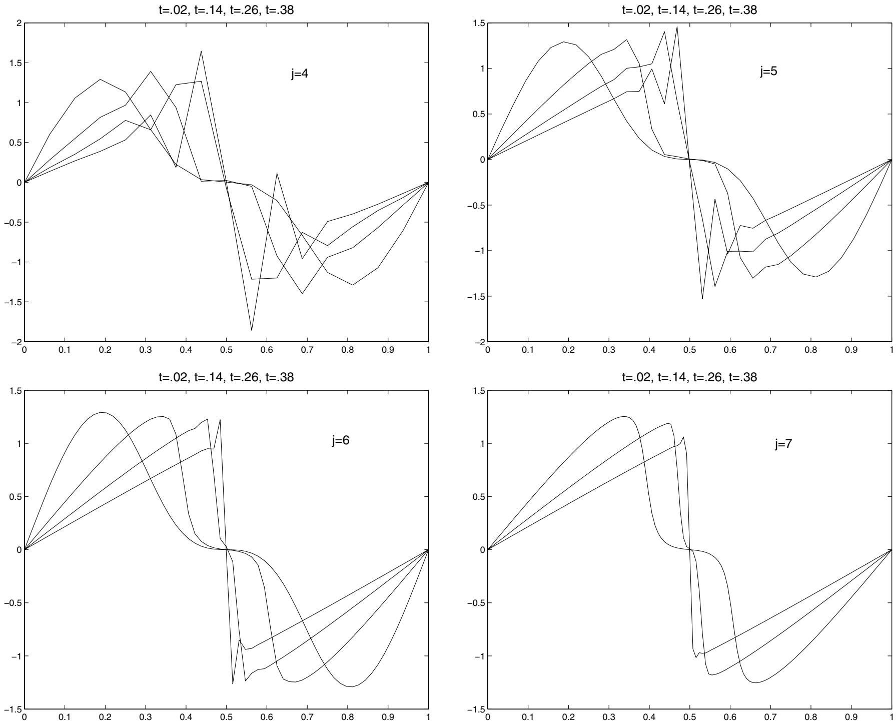


Figure 6.5: Numerical solution of the Burgers equation at various times and scales (j). The time step is $\Delta t = 10^{-4}$.

6.2 Generalized Burgers equation.

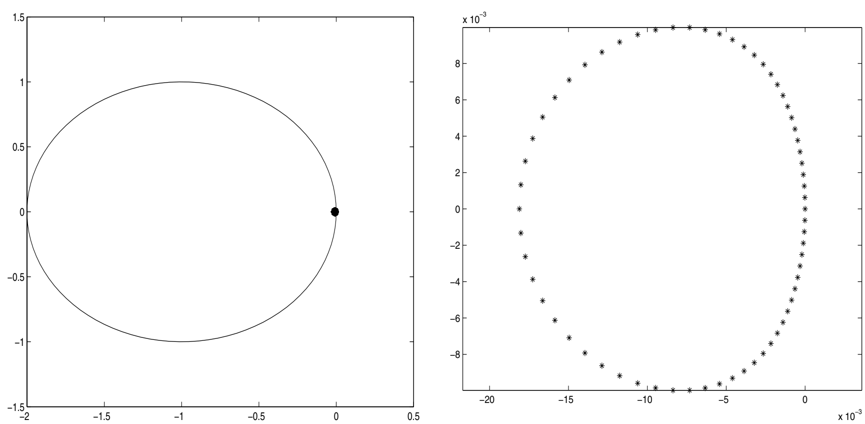
Here we consider the numerical solution of the generalized Burgers equation (3.8) for constant $\alpha, \beta = 1$, and $\lambda = -1$ and consider the evolution of a Gaussian initial condition centered on the interval $0 \leq x \leq 1$, e.g., $u(x,0) = e^{-(\sigma(x-\frac{1}{2}))^2}$ and periodic boundary conditions $u(x,0) = u(x,1)$. Figure 6.7 illustrates the solution using a W-TGM scheme for the scale $j = 6$, and the region of absolute stability is plotted in Figure 6.8.

7 Conclusion.

In this work a space and time accurate scheme called the wavelet- Taylor Galerkin method (W-TGM) for the Burgers equation is introduced. W-TGM appears to be fundamentally implicit, thereby demanding matrix inversion at each time marching step. To avoid this time-consuming step, explicit schemes based on approximate factorization is employed in solving the linear systems. The solution by these easy-to-implement and computationally economical explicit schemes is found to be quite close to the solution obtained by matrix inversion.

Table 6.2

j	Δt	at time:	WGM	W-TGM	W-TGM(explicit)	W-TGMS
4	10^{-4}	$t = 0.02$	0.0248	0.0248	0.0249	0.0248
		$t = 0.14$	0.1704	0.1692	0.1691	0.1691
		$t = 0.26$	0.2641	0.2626	0.2631	0.2625
		$t = 0.38$	0.9199	0.9183	0.9170	0.9180
5	10^{-4}	$t = 0.02$	0.0097	0.0096	0.0099	0.0096
		$t = 0.14$	0.0669	0.0650	0.0650	0.0650
		$t = 0.26$	0.2596	0.2567	0.2560	0.2563
		$t = 0.38$	0.5525	0.5532	0.5502	0.5522
6	10^{-4}	$t = 0.02$	0.0032	0.0031	0.0036	0.0031
		$t = 0.14$	0.0176	0.0163	0.0248	0.0164
		$t = 0.26$	0.1348	0.1319	0.1681	0.1313
		$t = 0.38$	0.2898	0.2899	0.2813	0.2878
7	10^{-4}	$t = 0.02$	0.0009	0.0009	0.0018	0.0009
		$t = 0.14$	0.0048	0.0048	0.0039	0.0047
		$t = 0.26$	0.1537	0.1513	0.3434	0.1508
		$t = 0.38$	0.0872	0.0873	0.0647	0.0874



(a) Absolute stability region for forward Euler and Δt times eigenvalues of L_6.

(b) Δt times eigenvalues of L_6.

Figure 6.6: Absolute stability region of the Burgers equation for case 2 where $\Delta t = 10^{-4}$, $\alpha = 1$, and $\nu = 10^{-2}/\pi$.

The wavelet-Taylor Galerkin splitting-up method (W-TGMS) for the Burgers equation works well and produces results nearly as accurate as W-TGM and W-TGM (explicit). The study motivates further research in developing W-TGM-

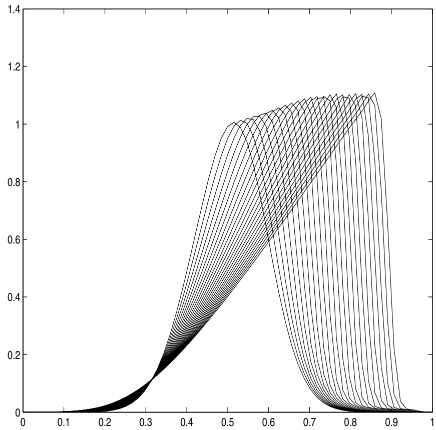


Figure 6.7: Solution at various time steps for $D = 6$, where $\Delta t = 0.005$ and $\nu = 0.005$.

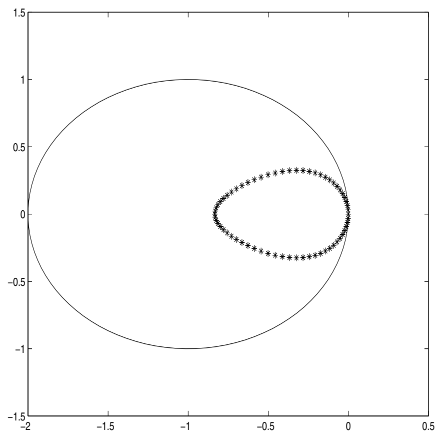


Figure 6.8: Absolute stability region for forward Euler and Δt times the eigenvalues of L_6 for $D = 6$, where $\Delta t = 0.005, \alpha = 1$, and $\nu = 0.005$.

based space–time-accurate schemes for multidimensional nonlinear systems like Navier–Stokes equations.

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