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Time-accurate solution of advection–diffusion problems by wavelet-Taylor–Galerkin method

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SUMMARY

In this paper we propose a wavelet Taylor–Galerkin method for the numerical solution of time-dependent advection–diffusion problems. The discretization in time is performed before the spatial discretization by introducing second- and third-order accurate generalization of the standard time stepping schemes with the help of Taylor series expansions in time step. Numerical schemes taking advantage of the wavelet bases capabilities to compress both functions and operators are presented. Numerical examples demonstrate the efficiency of our approach. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: Taylor–Galerkin method; wavelets; time marching scheme

1. INTRODUCTION

The application of methods based on wavelets to the numerical solution of partial differential equations (PDEs) has recently been studied both from the theoretical and the computational point of view due to its attractive feature: orthogonality, arbitrary regularity, good localization. Wavelet bases seem to combine the advantages of both spectral and finite element basis. Schematically the wavelet based methods for PDEs can be separated into three classes.

In a first class, wavelets are used, in the framework of a classical grid adaptive numerical code, to detect where the grid has to be refined or coarsed to optimally represent the solution. Instead of expanding the solution in terms of wavelets, the wavelet transform is used to determine the adaptive grid [1].

In a second class multiresolution analysis and their associated scale function bases may be used as alternative bases in Galerkin methods [2–4]. Such methods have thus convergence properties similar to the ones of spectral methods, and simultaneously partial derivative operators discretize similarly as in finite difference (FD) methods. However, as these methods do

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not use wavelets but rather scale function as basis functions, they cannot be adaptive methods and cannot reduce significantly the number of degree of freedom in a numerical code.

The third class, the only one which uses the compression properties of wavelet bases, contain specific wavelet methods for PDEs. In the literature, many tentatives have been performed, often based on Galerkin or Petrov–Galerkin methods. Some of them take advantage of the wavelet compression of the solution [5], this useful property also leads to the concept of adaptivity where the necessary information about how and where to spend the degrees of freedom has to be acquired during the solution process. Adaptive numerical concept for solving a wide class of variational problems have been recently studied in References [6–9]. Others use instead the wavelet compression of the operator [10]. The aim of the present paper is to introduce the wavelet-Taylor–Galerkin method (W-TGM) which has the benefit of both these two properties. The fundamental concept behind the Taylor–Galerkin approach [11, 12] is to incorporate more analytical information into the numerical scheme in the most direct and natural way, so that the technique may be regarded as an extension to PDEs of the Obrechkoff methods [13] for ordinary differential equations. Time accurate solution of Korteweg-de Vries equation using wavelet Galerkin method (WGM) is also developed in Reference [14] and wavelet multilayer Taylor–Galerkin schemes for hyperbolic and parabolic problems are introduced in Reference [15].

In this paper higher order accurate versions of the Crank–Nicolsan (CN) time stepping algorithms are developed on the basis of Taylor series expansion where the time derivatives are evaluated from the governing equation and where we are taking the advantage of sparsity of matrices which are coming in evolutionary problems. It can be generalized to any time stepping scheme based on Taylor series expansion. Our W-TGM is based on fast algorithms like matrix vector product in wavelet bases and wavelet compression property of smooth data.

Spatial approximation can be made by using different wavelet basis such as orthogonal Daubechies wavelets [16], biorthogonal spline wavelets [17], interpolats [18], etc. Our method works with any of these basis functions. In this paper we demonstrate our method using Daubechies compactly supported wavelets.

2. W-TGM FOR EVOLUTIONARY PROBLEMS

In the following, we give a brief introduction to wavelets and our notation used. We first deal with one-dimensional wavelets and then consider two variants for its generalization to the multivariate case. Finally, we describe how compression properties of wavelets can be used in W-TGM scheme for advection–diffusion problem in one dimension and hill translating, hill rotating around the origin in two dimension.

2.1. Univariate wavelets

The class of compactly supported wavelet bases was introduced by Daubechies in 1988. They are an orthonormal bases for functions in $L^2(R)$. A 'wavelet system' consists of the function $\phi(x)$ and the function $\psi(x)$ referred to as wavelet function. We define translates of $\phi(x)$ as

$$\phi_i(x) = \phi(x - i) \tag{1}$$

Multiresolution analysis (MRA) is the theory that was used by Ingrid Daubechies to show that for any non-negative integer n there exists an orthogonal wavelet with compact support such that all the derivatives up to order n exist. MRA describes a sequence of nested approximation spaces V_j in $L^2(R)$ such that closure of their union equals $L^2(R)$. MRA is characterized by the following axioms:

$$
\begin{gathered}
\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(R) \\
\overline{\bigcup_{j=-\infty}^{j=\infty} V_j} = L^2(R) \\
\bigcap_{j \in Z} V_j = 0 \\
f \in V_j \text{ if and only if } f(2(.)) \in V_{j+1} \\
\phi(x-k)_{k \in Z} \text{ is an orthonormal basis for } V_0
\end{gathered}
\tag{2}
$$

We define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e. $V_j \perp W_j$ and

$$V_{j+1} = V_j + W_j \tag{3}$$

$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)_{k \in Z}$ is an orthonormal basis for V_j and ϕ is the solution of the so-called scaling equation

$$\phi(x) = \sqrt{2} \sum_{k=0}^{D-1} a_k \phi(2x-k) \tag{4}$$

with explicitly known coefficients a_k (low pass filter). An analytical description of ϕ is not available, but it is also not needed. Wavelets are also dilates/translates of a single function ψ such that $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)_{k \in Z}$ is an orthonormal basis for W_j. Each member of the wavelet family is determined by the dilation equation

$$\psi(x) = \sqrt{2} \sum_{k=0}^{D-1} b_k \phi(2x-k) \tag{5}$$

where D is the order of wavelet and $b_k = (-1)^k a_{D-1-k},\ k = 0, 1, \ldots, D-1$. As pointed out by Meyer (1990) the complete toll box built in $L^2(R)$ can be used in the periodic case $L^2([0,1])$ by introducing a standard periodization technique. This technique consists at each scale in folding, around the integer values, the wavelet $\psi_{j,k}$ and the scaling functions $\phi_{j,k}$ centred in $[0,1]$. It writes $\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n)$ and $\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n)$ and generates V_{Pj} and W_{Pj}. A function $f \in V_{Pj}$ in pure periodic scaling function expansion $f(x) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_{j,k}(x)$ and the periodic wavelet expansion $f(x) = \sum_{k=0}^{2^{J_0}-1} c_k^{J_0} \tilde{\phi}_{J_0,k}(x) + \sum_{j=J_0}^{J-1} \sum_{k=0}^{2^j-1} d_k^j \tilde{\psi}_{j,k}(x)$, where J_0 satisfy $0 \leqslant J_0 \leqslant J$ and the decay of the wavelet coefficient is given by the following theorem [19].

Theorem
Let $P = D/2$ be the number of vanishing moments for a wavelet $\psi_{j,k}$ and let $f \in C^P(R)$. Then the wavelet coefficients decay as $|d_{j,k}| \leqslant C_P 2^{-j(P+1/2)} \max_{\xi \in I_{j,k}} |f^{(P)}(\xi)|$.

2.2. *Time dependent advection–diffusion problem*

Consider the linear advection–diffusion equation

$$\partial_t u = -a\partial_x u + v\partial_x^2 u \tag{6}$$

where a and $v > 0$ are positive constant coefficients.

2.2.1. Time discretization. We have used second-order W-TGM because too many terms are introduced in the third-order time derivative term, especially for non-linear problems. This difficulty may be circumvented by the use of a splitting-up method in which the advection–diffusion problem is decomposed into a pure advection problem followed by a pure diffusion problem, where the advection phase may be treated by third-order W-TGM. Let us leave the spatial variable x continuous and discretize (6) in time by the following forward Taylor series expansion. To obtain an improved order of accuracy in Δt we shall apply a Taylor–Galerkin method based on the following Taylor series expansions:

$$u^{n+1} = u^n + \delta t u_t^n + \frac{\delta t^2}{2} u_{tt}^n + \cdots \tag{7}$$

$$u^n = u^{n+1} - \delta t u_t^{n+1} + \frac{\delta t^2}{2} u_{tt}^{n+1} + \cdots \tag{8}$$

Combination of these two gives

$$\frac{u^{n+1} - u^n}{\delta t} = \frac{1}{2}(u_t^n + u_t^{n+1}) + \frac{\delta t}{4}(u_{tt}^n - u_{tt}^{n+1}) \tag{9}$$

replacing the time derivatives by spatial derivatives, the associated wavelet-Taylor–Galerkin equations based on CN time stepping scheme are which includes first- and second-order time derivatives. while the former is provided directly by (6), the latter can be obtained by taking the time derivative of the governing PDEs. The time derivative of (6) is

$$u_{tt} = -a^2\partial_x^2 u - 2av\partial_x^3 u + v^2\partial_x^4 u \tag{10}$$

and the substitution of (6) and (10) into the Taylor series expansion (9) gives W-TGM scheme

$$Au^{n+1} = Bu^n \tag{11}$$

where $A = I - \delta t/2(-a\partial_x + v\partial_x^2) + \delta t^2/4(-a^2\partial_x^2 - 2av\partial_x^3 + v^2\partial_x^4)$ and $B = I + \delta t/2(-a\partial_x + v\partial_x^2) + \delta t^2/4(-a^2\partial_x^2 - 2av\partial_x^3 + v^2\partial_x^4)$. Now wavelet Galerkin discretization turns the problem into a finite dimensional space

$$d_u^{n+1} = \mathscr{A}^{-1}\mathscr{B}d_u^n = \mathscr{D}d_u^n \tag{12}$$

In this finite dimensional space u^n is to be replaced by the vector d_u^n along a wavelet finite basis, and A and B are replaced by $\mathscr{A}$ and $\mathscr{B}$ (finite) matrices, respectively. Due to second- and third-order term in Taylor series our scheme leads to implicit method that needs inversion. Now to solve Equation (11) in wavelet basis we will compute $\mathscr{A}^{-1}$ and $\mathscr{A}^{-1}\mathscr{B}$ once and

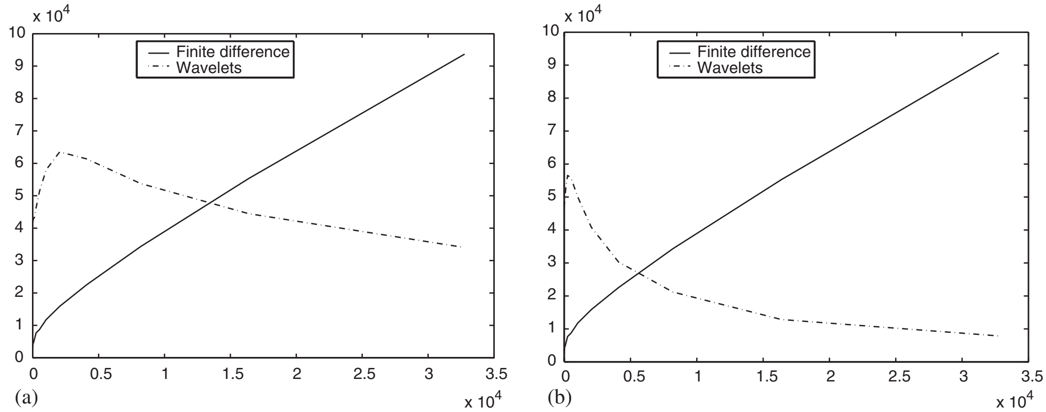

Figure 1. Number of coefficients in the successive powers D^n: (a) CN times stepping in wavelets and in finite difference, versus $x=2^n$, $n=15$, $N=1024$, $v\delta t=10^{-5}$, $\varepsilon_M=10^{-8}$; and (b) Taylor–Galerkin approach in wavelets and in finite differences.

store in compressed form. We can now give a computational procedure for computing (11) using wavelet compression.

Algorithm

1. $\text{trunc}(\mathscr{A}^{-1},\varepsilon_M) \dashrightarrow (\mathscr{A}^{-1})^{\varepsilon_M}$,
2. compute initial guess in wavelet basis $\dashrightarrow d_u^0$,
3. $\text{trunc}(d_u^0,\varepsilon_V) \dashrightarrow (d_u^0)^{\varepsilon_V}$
   for $n=0,1,\ldots,n1-1$,
4. $(\mathscr{A}^{-1})^{\varepsilon_M}\mathscr{B}(d_u^n)^{\varepsilon_V} \dashrightarrow d_u^{n+1}$,
5. $\text{trunc}(d_u^{n+1},\varepsilon_V) \dashrightarrow (d_u^{n+1})^{\varepsilon_V}$,
   where $\text{trunc}(d_u,\varepsilon_V)=\{d_k^j, \quad |d_k^j|>\varepsilon_V\}$ and $\text{trunc}(\mathscr{A},\varepsilon_M)=\{[\mathscr{A}_{m,n}] \quad [\mathscr{A}_{m,n}]>\varepsilon_M\}$.

A further property of the wavelet representation of operators is that the successive powers $\mathscr{D}^n$ of the time iteration matrix become sparser and sparser with increasing n. This property is very specific to wavelets, as the opposite occurs with finite difference where $\mathscr{D}^n$ becomes a more and more dense matrix as shown in Figure 1. It is seen from 1 that in wavelet-Taylor–Galerkin approach compression in the matrix D^n is larger than wavelet Galerkin approach. From this property we can obtain iterative speed of the wavelet-Taylor–Galerkin scheme.

1. Initialize $(\mathscr{A}_0^{-1})^{\varepsilon_M}$ and $(d_u^0)^{\varepsilon_V}$,
2. $(\mathscr{D}_0)^{\varepsilon_M} \dashrightarrow (\mathscr{A}_0^{-1})^{\varepsilon_M}\mathscr{B}$
   for $n=0,1,\ldots,n1-1$,
3. $(\mathscr{D}_n)^{\varepsilon_M}(d_u^n)^{\varepsilon_V} \dashrightarrow (d_u^{n+1})^{\varepsilon_V}$,
4. $\mathscr{D}_n^2 \dashrightarrow \mathscr{D}_{n+1}$. Then the approximate solution of PDE is at $t=2^n\delta t$ is $d_u^{(2^n)}$.

Since differential operators are local operators, it seems that not much can be gained by compression. But in wavelet basis it is possible to efficiently invert the differential operator and then approximate (in a compressed form) the dense evolution operators. There is no need to change from classical to wavelet co-ordinates till some time steps. In classical co-ordinate, the evolution operator changes from very sparse to dense. In the wavelet representation we may start the squaring in the classical co-ordinates and change to the wavelet basis at the point where the wavelet representation is sparser. Thus, we have the following algorithm:

1. For $n=0,2,\ldots,p$,
2. $(A)^{-1}Bu^n \dashrightarrow u^{n+1}$,
3. Initialize $(\mathscr{A}^{-1})^{\varepsilon_V}$ and $(d_u^p)^{\varepsilon_V}$
   for $n=p+1,p+2,\ldots,p+n1-1$,
4. $(\mathscr{A}^{-1})^{\varepsilon_M}\mathscr{B}(d_u^n)^{\varepsilon_V} \dashrightarrow (d_u^{n+1})$,
5. $\mathrm{trunc}(d_u^{n+1},\varepsilon_V) \dashrightarrow (d_u^{n+1})^{\varepsilon_V}$.

It is essential for the success of this algorithm that the computation of the matrix vector product fully exploits the compressed form of both matrix and vectors. This can be done using the algorithm in Reference [19] or fast multiplication based on a general sparse format for both matrix and vector as done in Reference [20]. This novel and important element of computing in wavelet bases is that the compressed operators in standard and non-standard form can be multiplied rapidly. The product of two operators in the standard form requires $C(-\ln\varepsilon)N$ and multiplication of operator in non-standard form requires order $(-\ln\varepsilon)N$ operations where ε is desired accuracy. This implicit wavelet Galerkin schemes are competitive only if inverse can be computed efficiently. Due to our periodic domain inverse of evolution operator can be efficiently computed by FFT technique as described in Reference [19]. Another technique to invert them efficiently directly in wavelet basis by using a iteration algorithm described by Beylkin [20] which requires fast matrix multiplication algorithm as described above. We are using this technique.

Another wavelet-Taylor–Galerkin scheme can also be formulated from other time stepping, i.e. leap-frog, forward Euler, etc. In all these methods a fundamental role is played by the Taylor series in the time increment which is exploited indirectly in multi-step schemes and directly in single step ones. In this respect, the two different classes of methods correspond to Runge–Kutta and Obrechkoff [13] methods, respectively, for ordinary differential equations. W-TGM scheme has the inconvenience in using the higher order time derivative for calculating non-linear problems. Therefore, we can also use the following Runge–Kutta form of Lax–Wendroff scheme. By approximating Equation (7) up to third-order accuracy, the formulation of this scheme can be written as

$$\begin{aligned} u^{k+1/3} &= u^k + (\delta t/3)u_t^k \\ u^{k+1/2} &= u^k + (\delta t/2)u_t^{k+1/3} \\ u^{k+1} &= u^k + \delta t u_t^{k+1/2} \end{aligned} \tag{13}$$

After putting time derivative from the governing PDEs spatial discretization of Equation (13) can be performed by WGM. To deal with transient situations which evolve toward a highly convective state, the global Taylor–Galerkin method is found to be ineffective since it reduces to standard Galerkin method as temporal term vanishes and to avoid highly complicated terms which comes from the diffusion term after applying third-order W-TGM scheme, we can use operator splitting as done in paper [14].

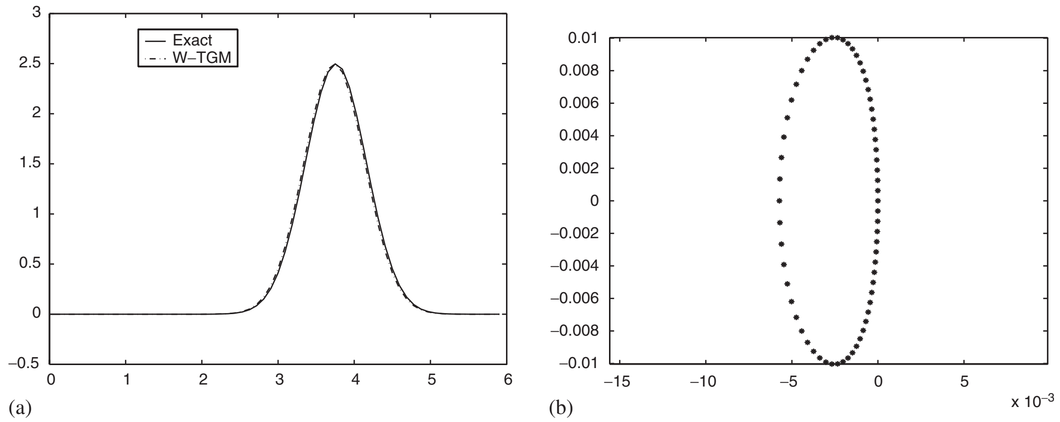

Figure 2. (a) W-TGM solution vs exact solution for advection–diffusion problem; and (b) real and imaginary parts of ($\delta t *$ eigenvalues) computed for asymptotic stability analysis of W-TGM on advection–diffusion problem.

2.2.2. Theoretical stability of the linearized schemes. We use the notion of asymptotic stability of a numerical method as it is defined in References [13, 21] for a discrete problem of the form $\mathrm{d}u/\mathrm{d}t = Lu$ where L is assumed to be diagonal matrix. The region of absolute stability of a numerical method is defined for the scalar model problem $\mathrm{d}u/\mathrm{d}t = \lambda u$ to be set of all $\lambda\delta t$ such that $\|u^n\|$ is bounded as $t \longrightarrow \infty$. Finally, we say that a numerical method is asymptotic stable for a particular problem if, for small $\delta t > 0$, the product of δt times every eigenvalues of L lies within the region of absolute stability. In Equation (9) putting $u_t^n = \lambda u^n$ and $u_{tt}^n = \lambda^2 u^n$ we will get the following equation:

$$\left(1 - \frac{\lambda\delta t}{2} + \frac{\lambda^2\delta t^2}{4}\right) u^{n+1} = \left(1 + \frac{\lambda\delta t}{2} + \frac{\lambda^2\delta t^2}{4}\right) u^n \tag{14}$$

So for the numerical stability of the scheme δt should satisfy the following condition:

$$\left|1 + \frac{\lambda\delta t}{2} + \frac{\lambda^2\delta t^2}{4}\right| < \left|1 - \frac{\lambda\delta t}{2} + \frac{\lambda^2\delta t^2}{4}\right| \tag{15}$$

From this condition we can conclude the absolute stability region of second-order accurate W-TGM will be entire left half plane.

2.2.3. Numerical results. The accuracy of the proposed W-TGM has been verified numerically on the classical test problem of advection–diffusion of a Gaussian profile. The exact solution is $u(x,t) = (1/\sigma(t))\exp[-(x - x_0 - at)^2/2\sigma(t)^2]$, where $\sigma(t) = \sigma_0(1 + 2vt/\sigma_0^2)^{1/2}$. The parameters are given by $x_0 = 3.75$, $a = 1$, $v = 0.01$. Figure 2(a) shows the comparison of numerical solution obtained for $\delta t = 10^{-3}$ using $D6$ scaling and wavelet function (D6 stands for Daubechies wavelet which has a support in $[0, 5]$) with the exact solution, Figure 2(b) shows the δt times the eigenvalues of matrix resulting from W-TGM scheme. Due to complex eigenvalues it is plotted in complex domain. The stability region of Figure 2(b) always satisfy the stability criteria. Figure 3 shows the exact solution and numerical results obtained by different

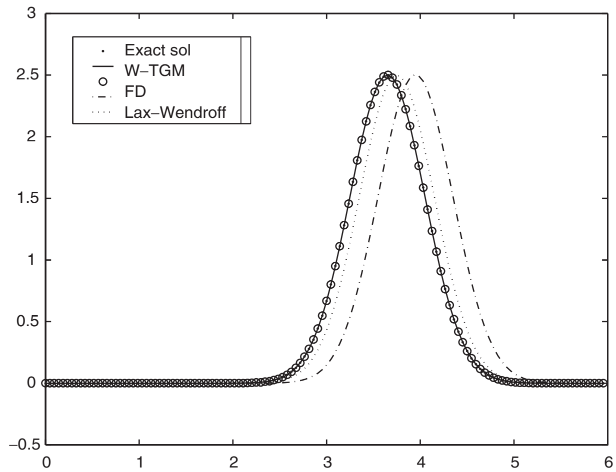

Figure 3. Solution at $t = 0.1$ with $\delta t = 10^{-3}$.

Table I. Compression error for WGM scheme.

$\varepsilon_V = 0$ ε_M	% elem $(\mathscr{A}^{-1})^{\varepsilon_M}$	$E^{\varepsilon_M,\varepsilon_V}$	$\varepsilon_M = 0$ ε_V	% elem $(d_u^n)^{\varepsilon_V}$	$E^{\varepsilon_M,\varepsilon_V}$
10^{-10}	14.74	7.2e − 11	10^{-10}	61.78	5.6e − 10
10^{-9}	14.32	2.2e − 09	10^{-9}	57.81	5.3e − 09
10^{-8}	13.58	8.64e − 09	10^{-8}	56.25	4.5e − 08
10^{-7}	13.18	2.3e − 06	10^{-7}	51.56	3.1e − 07
10^{-6}	11.79	2.21e − 05	10^{-6}	49.21	3.8e − 06

method. Here we are using W-TGM based on Euler time stepping. It is clear that W-TGM is giving more accurate solution compared to other methods. The vector $u^{\varepsilon_M,\varepsilon_V}$ is the computed solution given the threshold ε_M and ε_V. Hence, we define the relative compression error as

$$E^{\varepsilon_M,\varepsilon_V} = \frac{\|u^{\varepsilon_M,\varepsilon_V} - u^{0,0}\|_\infty}{\|u^{0,0}\|_\infty}$$

Table I shows the relative error introduced by compression $E^{\varepsilon_M,\varepsilon_V}$. It is seen from Figure 1 that significant compression is achieved in matrix $\mathscr{D}^n$ and in wavelet-Taylor–Galerkin approach number of elements in matrix $\mathscr{D}^n$ is decaying faster than wavelet Galerkin approach. Here significant compression is also achieved in solution vector (Table II).

Secondly we are showing the power of W-TGM on the problem of advection–diffusion equation with $u_0(x) = \sin(x)$ where $x \in [0, 2\pi]$ with the analytical solution $u(x,t) = e^{-vt} \sin(x - at)$. Here we are using W-TGM based on Euler time stepping. In this case $A = I - (\delta t/2)(-a\partial_x + v\partial_x^2)$ and $B = I + (\delta t/2)(-a\partial_x + v\partial_x^2) + (-a\partial_x + v\partial_x^2)$. The comparison of errors resulting in FD computation, WGM and W-TGM with Euler time stepping with respect to the exact solution in L_∞ norm are shown in Table III for the parameters $a = 1$, $v = 0.001$, $\delta t = 10^{-2}$ with different value of j for $D6$ wavelet.

Table II. Compression error for W-TGM scheme.

$\varepsilon_V = 0$ ε_M	% elem $(\mathscr{A}^{-1})^{\varepsilon_M}$	$E^{\varepsilon_M,\varepsilon_V}$	$\varepsilon_M = 0$ ε_V	% elem $(d_u^n)^{\varepsilon_V}$	$E^{\varepsilon_M,\varepsilon_V}$
10^{-10}	18.31	8.6e − 11	10^{-10}	88.28	6.6e − 10
10^{-9}	17.64	6.2e − 10	10^{-9}	85.16	2.4e − 09
10^{-8}	16.88	4.6e − 9	10^{-8}	80.47	3.3e − 08
10^{-7}	15.81	1.3e − 07	10^{-7}	74.22	3.3e − 07
10^{-6}	13.61	2.4e − 05	10^{-6}	67.19	4.7e − 06

Table III. Comparison of errors for FD, WGM, W-TGM for advection–diffusion problem.

j	FD	WGM	W-TGM
5	0.8654	1.499e − 004	1.7033e − 007
6	0.8654	1.492e − 004	2.7357e − 008
7	—	1.534e − 5	2.5026e − 008
9	—	1.123e − 5	2.4891e − 008

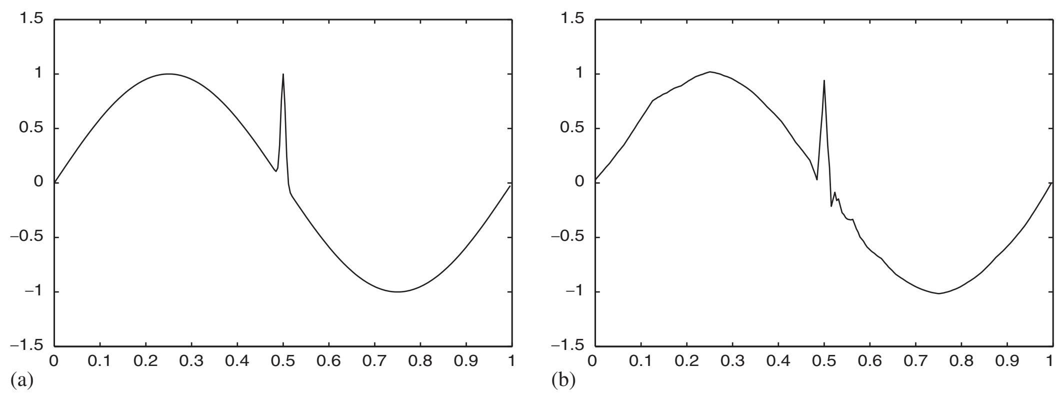

Figure 4. (a) Initial function $u_0(x)$; and (b) truncated approximate initial solution by D_6 with the threshold $\varepsilon = 0.001$, which leads to 13 retained wavelet coefficients, out of original 2^8.

Further a '—' in the table denotes that the scheme is unstable.

To show the power of wavelet by W-TGM scheme here we are considering advection–diffusion equation with initial condition $u_0(x) = \sin(2\pi x) + \exp^{-\alpha(x-1/2)^2}$, which is smooth in most of the domain except near $x = 0.5$ where we have a spike and the thresholded wavelet expansion of the initial solution $u(x, 0)$ is shown in Figure 4. This wavelet expansion will have few coefficients, except for in the neighbourhood of the spike at $x = 0.5 - t$. In terms of FD methods, we want to have many points in areas where the solution has strong variation and few points in area where the solution is smooth. If we use a Galerkin method, this corresponds to the representation of the solution having fewer basis functions in the smooth

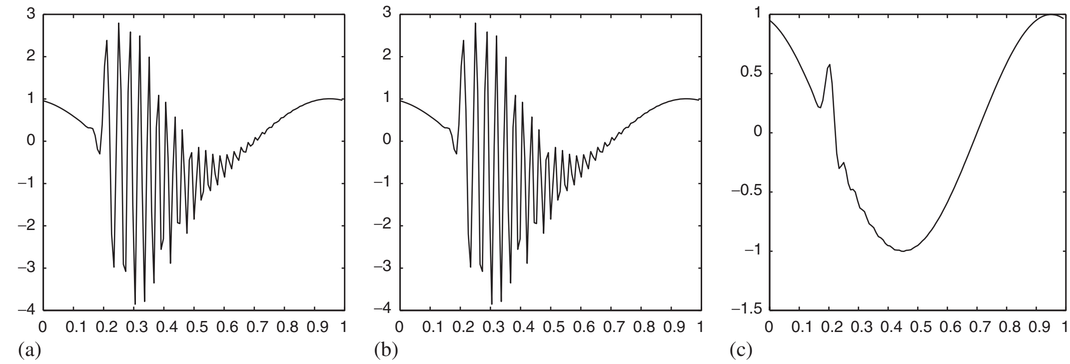

Figure 5. Solution of advection equation at $t = 0.3$, $v = 0.0001$, $j = 7$: (a) FD; (b) WGM; and (c) W-TGM.

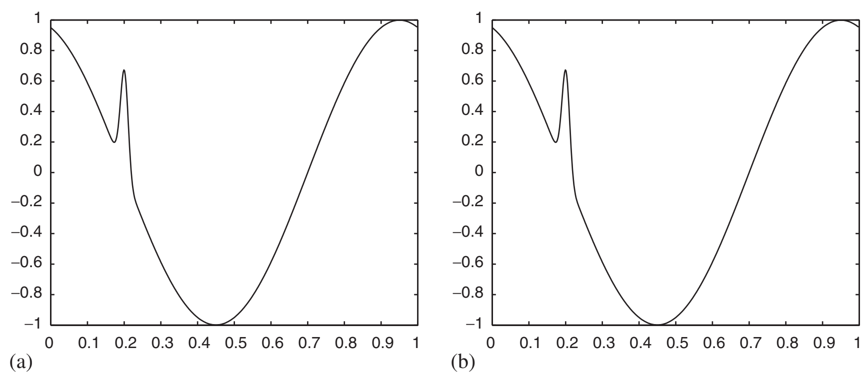

Figure 6. Solution of advection equation at $t = 0.3$, $v = 0.0001$, $j = 9$: (a) W-TGM, $\varepsilon_M = 0$, $\varepsilon_v = 0$; and (b) W-TGM, $\varepsilon_M = 0$, $\varepsilon_v = 10^{-6}$.

areas. Note that by thresholding a wavelet representation we have a way to automatically find a sparse representation of smooth part. Holmstrom and Walden have applied adaptive wavelet methods on such type of PDEs [22, 23]. We stepped forward to $t = 0.3$, where the solution for $j = 7$ is shown in Figure 5 using FD, WGM and W-TGM based on Euler time stepping. In FD and WGM scheme oscillation are growing very fast with the increasing value of j. For $j = 7$ improvement of accuracy for W-TGM scheme over FD and WGM can be seen from Figure 5. For $j = 8, 9$ FD and WGM is completely giving unstable solution due to oscillation near the spike, whereas in our approach of W-TGM scheme we are taking the advantage of time accurate scheme as well as wavelet capabilities of compression to produce fast algorithm based on fast matrix vector product in terms of sparsity. We are getting oscillation free solution as shown in Figure 6 without and with truncation. FD and WGM are unable to produce oscillation free solution near the spike with the same degree of freedom. For higher value of $v = 0.001$ solution is shown in Figure 7 using W-TGM scheme where both FD and WGM scheme is producing unstable result.

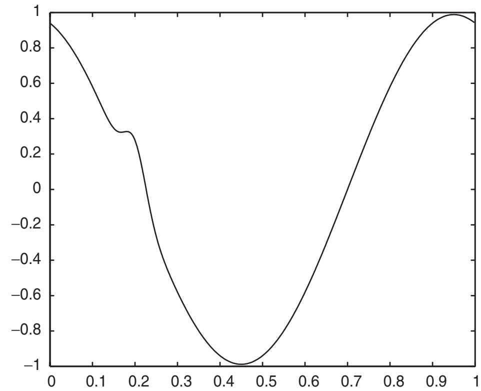

Figure 7. Solution of advection equation at $t = 0.3$, $v = 0.001$.

2.3. Multi-dimensional problems

In order to show that the proposed W-TGM is applicable to multidimensional situations, we need a way to define multivariate wavelets. The simplest way to obtain multivariate wavelets is to employ anisotropic or isotropic tensor products. We have used second approach. To show that this is effective, we shall consider the standard test problems of hill translation and hill rotation [24].

Case 1: The problem of a Gaussian hill translating with a uniform velocity a and spreading isotropically with a diffusivity v is governed by

$$u_t = -a.\nabla u + v\nabla^2 u \tag{16}$$

Here time discretization will be same as in one-dimensional case as in Equation (9) second-order W-TGM scheme. The equations are integrated till time $t = 0.5$ is reached. The initial distribution and the solution at $t = 0.5$ without and with compression are shown in Figure 8.

Case 2: Consider the problem of a hill rotating around the origin governed by

$$\partial_t u + \nabla.\begin{pmatrix} -yu \\ xu \end{pmatrix} = v\Delta u \tag{17}$$

Here we are using second-order accurate W-TGM scheme based on following Taylor series expansion:

$$\frac{u^{n+1} - u^n}{\delta t} = \frac{1}{2}(u_t^n + u_t^{n+1}) + \frac{\delta t}{4}(u_{tt}^n - u_{tt}^{n+1}) \tag{18}$$

Let $\nabla.\binom{-yu}{xu} = v.\nabla u$ then the original equation is $u_t = -(v.\nabla u) + v\Delta u$ and time derivative is $u_{tt} = v.\nabla(v.\nabla u) - vv.\nabla(\Delta u) + v\Delta(-(v.\nabla u + v\Delta u))$, putting these values in Equation (18) till second-order term, we will get a matrix form

$$Au^{n+1} = Bu^n \tag{19}$$

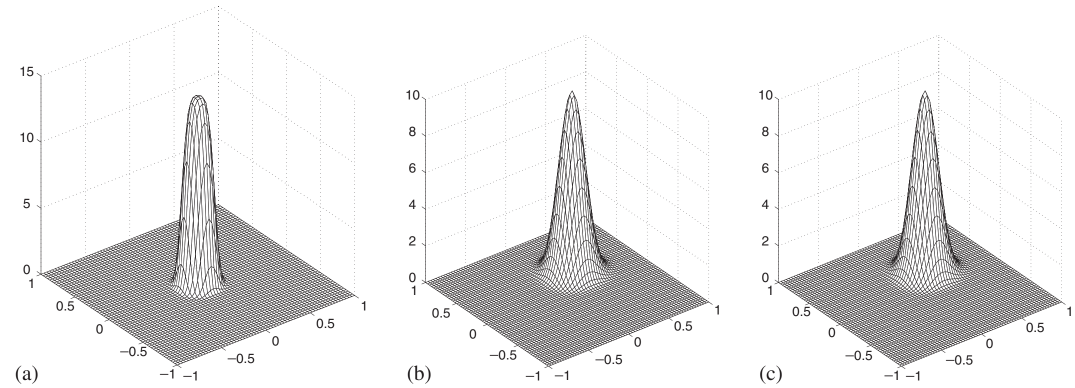

Figure 8. (a) Initial distribution of the hill; (b) solution at $t=0.5$ without compression for W-TGM scheme; and (c) solution at $t=0.5$ with compression for W-TGM scheme.

To avoid higher order derivatives of diffusion term we can use operator splitting as done in Reference [14]. Unfortunately for a $N\times N$ space discretization, this will result in a system of algebraic equation that is sparse but is of the order $N^2\times N^2$. To overcome this problem, we are taking full advantage of tensorial wavelet bases that leads to the factorization approximation of 2D hill rotating operator.

$$\text{For small } \alpha \text{ and } \beta(I-\alpha\Delta+\beta\Delta^2)\approx\left(I-\alpha\frac{\partial^2}{\partial x^2}+\beta\frac{\partial^4}{\partial x^4}\right)\left(I-\alpha\frac{\partial^2}{\partial y^2}+\beta\frac{\partial^4}{\partial y^4}\right) \tag{20}$$

Then our W-TGM scheme express themselves as products of $N\times N$ matrices

$$d_u^{n+1}=Bd_u^nB^{\mathrm{T}}+\delta tA^{-1}d_fA^{-1^{\mathrm{T}}} \tag{21}$$

All the algorithms developed in one dimension using compression of wavelet can be generalized in two dimension also. To assess the accuracy of the W-TGM scheme in two-dimensional situations, first Equation (17) is considered with initial condition $u(x,0)=(1+\cos(2\pi R))/2$ for $R<0.5$, $u(x,0)=0$ otherwise, where $R=\sqrt{x^2+(y-0.5)^2}$. For $v=0$, initial data and numerical solution without and with truncation are shown in Figure 9. The solution is in good agreement with those presented by Smolianski and Kuzmin in Reference [25]. Secondly, Equation (17) is considered with initial condition $u(x,0)=\max(0,-3+4\mathrm{e}^{-30|x-(0.25,0.25)|^2})$ for non-zero value of v. Because of the small dissipative effects the solution is small at the boundaries. Therefore, periodic boundary conditions are used and $v=10^{-3}$. The result obtained by the W-TGM scheme is shown in Figure 10. Numerical scheme shows remarkable accuracy. In addition the final height of the cone was found 98 per cent of its original value, indicating a minimal dissipation error.

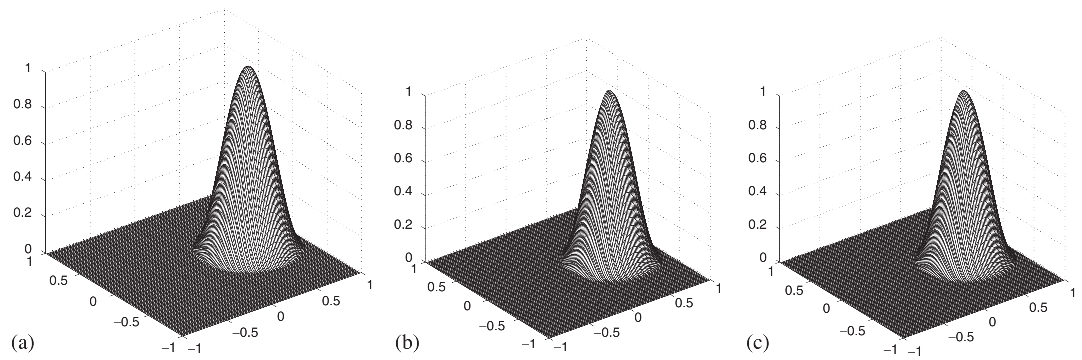

Figure 9. (a) Min $u = 0$, max $u = 1$; (b) min $u = -3.9 \times 10^{-6}$, max $u = 0.9926$, $\varepsilon_M = 0$, $\varepsilon_v = 0$; and (c) min $u = -10^{-6}$, max $u = 0.9926$, $\varepsilon_M = 10^{-5}$, $\varepsilon_v = 10^{-5}$.

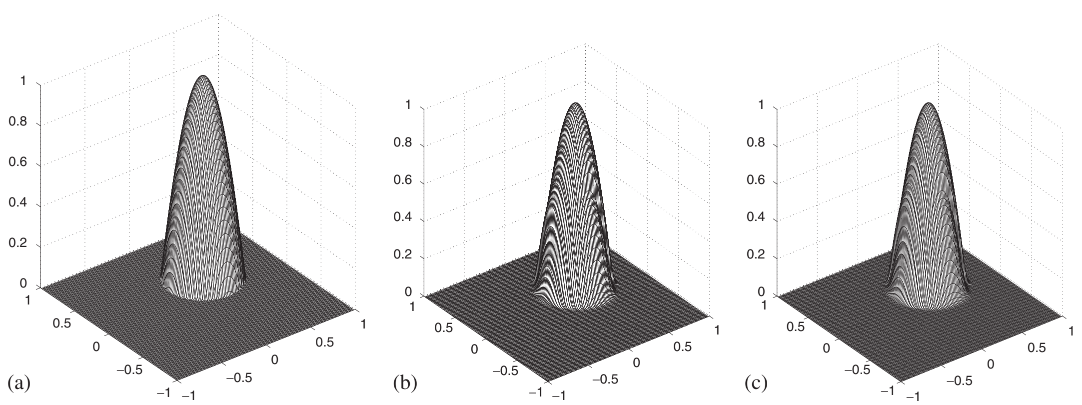

Figure 10. (a) Min $u = 0$, max $u = 1$; (b) min $u = -8.4 \times 10^{-5}$, max $u = 0.9887$, $\varepsilon_M = 0$, $\varepsilon_v = 0$; and (c) min $u = -8.4 \times 10^{-5}$, max $u = 0.9887$, $\varepsilon_M = 10^{-5}$, $\varepsilon_v = 10^{-5}$.

3. CONCLUSION

In this work a space and time accurate scheme called W-TGM for advection–diffusion problem in one- and two-dimension are introduced, where we can take the benefit of useful properties of wavelet for compression of operator and solution. W-TGM appears to be fundamentally implicit, thereby demanding matrix inversion at each time marching step. To avoid this time consuming step we are taking the advantage of wavelet compression and for two-dimension problem we are also using approximate factorization technique. The results are quite encouraging and we are extending our study to 3D and more complex phenomena of turbulence where spurious oscillations are more serious when a significant gradient is involved. This technique is easy to implement and computationally efficient.

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