International Journal of Computational Methods
Vol. 2, No. 1 (2005) 75–97

World Scientific
www.worldscientific.com

A WAVELET-TAYLOR GALERKIN METHOD FOR PARABOLIC AND HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

B. V. RATHISH KUMAR* and MANI MEHRA†

Department of Mathematics, Indian Institute of Technology
Kanpur — 208 016, India
**bvrk@iitk.ac.in*
†*manimeh@iitk.ac.in*

Received 16 April 2004
Accepted 21 September 2004

In this study a set of new space and time accurate numerical methods based on different time marching schemes such as Euler, leap-frog and Crank-Nicolson for partial differential equations of the form $u_t = \mathcal{L}u + \mathcal{N}f(u)$, where $\mathcal{L}$ is linear differential operator and $\mathcal{N}f(u)$ is a nonlinear function, are proposed. To produce accurate temporal differencing, the method employs forward/backward time Taylor series expansions including time derivatives of second and third order which are evaluated from the governing partial differential equation. This yields a generalized time discretized scheme which is approximated in space by Galerkin method. The compactly supported orthogonal wavelet bases developed by Daubechies are used in Galerkin scheme. This new wavelet-Taylor Galerkin approach is successively applied to heat equation, convection equation and inviscid Burgers' equation.

Keywords: Taylor-Galerkin method; wavelets; time marching scheme; parabolic equation; hyperbolic equation.

1. Introduction

In the past two decades interest in wavelets has been nothing short of being remarkable. Wavelet analysis assumed significance due to successful applications in signal and image processing during the eighties. The study of wavelets attained the present growth after the mathematical analysis of wavelets by Stromberg [1981], Grossmann and Morlet [1984] and Meyer [1985]. The multiresolution analysis of Mallat and Meyer [1989] led to Daubechies [1988] orthonormal family of wavelets. As wavelet theory progressed and more tools became available, their use spread to areas other than signal processing.

Recently wavelet Galerkin approximations have been studied as an alternative to conventional finite difference (FD) and finite element methods for partial differential equations (PDEs). The first applications of wavelets to the solution of PDEs

*Corresponding author.

seems to have consisted of Galerkin methods on problems with periodic boundary conditions as shown in [Amaratunga *et al.* (1994)]. Indeed, as was noted by Meyer [1990], the techniques which were first developed on the real line could be easily modified by a standard procedure of periodization to be used on $L^2([0,1])$ in the periodic case. Another approach into the study of wavelet differential operators is by Beylkin *et al.* [1991], Froleich *et al.* [1997] and Restrepo *et al.* [1995] found this approach is a viable option for the hyperbolic Boussinesq system of PDEs. Glowinski *et al.* [1990] considered wavelet based variational methods to solve one dimension linear and non-linear ordinary differential equations and observed that wavelets provide a robust and accurate alternative to more traditional methods. While the investigations of Liandrat *et al.* [1992] and Latto and Tenenbaum deal with periodic boundary conditions for the Burgers' equation. In another useful approach wavelets are used to drive adaptive finite difference method, as advocated by Jameson [1998] and later this issue has been established in a series of papers [Holmstrom (1999); Holmstrom and Walden (1998); Fatkulin and Hesthaven (2001)]. An adaptive second generation wavelet collocation method for evolution problems by Vasilyev is discussed in [Vasilyev and Bowman (2000); Vasilyev (2003)].

In the conventional numerical approach to transient problems the accuracy gained in using the high order spatial discretization is partially lost due to use of low-order time discretization schemes. Here usually spatial approximation precedes the temporal discretization. On the contrary, the reversed order of discretization can lead to better time accurate schemes with improved stability properties. Lax and Wendroff demonstrated this idea in the finite difference context [Lax and Wendroff (1960, 1962, 1964)] and has also been considered in conjunction with a spatial representation of spectral type [Gazdag (1973)]. Later Donea [1984, 1987] has used it in deriving a time accurate finite element scheme. Primarily their approach consists of extending the Taylor series in the time increment to the third order before spatial discretization. This procedure has not be implemented so far in the wavelet approach to evolutionary problems where again spatial approximation is known to precede the temporal discretization. In this paper we develop wavelet-Taylor Galerkin method (W-TGM) for a class of linear problems like heat equation, convective transport problem and non-linear hyperbolic conservation equation. We next making use of space-time relation as in given PDEs suitably express the temporal derivatives in spatial terms to propose a modified version of W-TGM called W-TGMS. Further exploiting the beneficial effect of the consistent mass matrix in a lumped-matrix context we also propose and implement a two-pass explicit version of W-TGM call it twoW-TGM. A time accurate solution of Korteweg-de Vries equation using wavelet Galerkin method is developed in [Kumar and Mehra].

The outline of paper is as follows. In Sec. 2 we summarize some basics of wavelet analysis. In Sec. 3 we introduce W-TGM, twoW-TGM, W-TGMS for the solution of time dependent PDEs. We describe this algorithm first for the heat equation and then for convection equation and inviscid Burgers' equation. In Sec. 4 we illustrate the accuracy by providing the results of W-TGM along with known methods like wavelet Galerkin method (WGM), Lax-Wendroff and comparing them with the exact solutions. In Sec. 5 we have studied numerically the stability of the algorithm in these cases. Finally in Sec. 6 we draw a number of conclusion based on our results and indicate directions of further investigation.

2. Wavelet Preliminaries

2.1. *Compactly supported wavelets*

The class of compactly supported wavelet bases was introduced by Daubechies in 1988. They are an orthonormal bases for functions in $L^2(R)$. A "Wavelet System" consists of the function $\phi(x)$ and the function $\psi(x)$ referred to as wavelet function. We define translates of $\phi(x)$ as

$$\phi_i(x) = \phi(x - i). \tag{1}$$

Multiresolution analysis (MRA) is the theory that was used by Ingrid Daubechies to show that for any non negative integer n there exists an orthogonal wavelet with compact support such that all the derivatives up to order n exist. MRA describes a sequence of nested approximation spaces V_j in $L^2(R)$ such that closure of their union equals $L^2(R)$. MRA is characterized by the following axioms

$$\begin{gathered}
\{0\} \subset \cdots \subset V_{-1} \subset V_0 \subset V_1 \cdots \subset L^2(R) \\
\overline{\bigcup_{j=-\infty}^{j=\infty} V_j} = L^2(R) \\
\bigcap_{j\in Z} V_j = 0 \\
f \in V_j \text{ if and only if } f(2(\cdot)) \in V_{j+1} \\
\phi(x-k)_{k\in Z} \text{ is an orthonormal basis for } V_0.
\end{gathered} \tag{2}$$

We define W_j to be the orthogonal complement of V_j in V_{j+1}, i.e. $V_j \perp W_j$ and

$$V_{j+1} = V_j + W_j, \tag{3}$$

$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k)_{k\in Z}$ is an orthonormal basis for V_j, $\psi_{j,k} = 2^{j/2}\psi(2^j x - k)_{k\in Z}$ is an orthonormal basis for W_j. Each member of the wavelet family is determined by the set of constants a_k (low pass filter) by the dilation equation

$$\phi(x) = \sqrt{2}\sum_{k=0}^{D-1} a_k \phi(2x - k), \tag{4}$$

and the equation

$$\psi(x) = \sqrt{2}\sum_{k=0}^{D-1} b_k \phi(2x - k), \tag{5}$$

where D is the order of wavelet and $b_k = (-1)^k a_{D-1-k},\ k = 0, 1, \ldots, D-1$.

As pointed out by Meyer [1990] the complete toll box built in $L^2(R)$ can be used in the periodic case $L^2([0,1])$ by introducing a standard periodization technique. This technique consists at each scale in folding, around the integer values, the wavelet $\psi_{j,k}$ and the scaling functions $\phi_{j,k}$ centered in [0,1]. It writes $\tilde{\phi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \phi_{j,l}(x+n)$ and $\tilde{\psi}_{j,l}(x) = \sum_{n=-\infty}^{\infty} \psi_{j,l}(x+n)$ and generates V_{Pj} and W_{Pj}. A function $f \in V_{PJ}$ in pure periodic scaling function expansion $f(x) = \sum_{k=0}^{2^j-1} c_k^j \tilde{\phi}_{j,k}(x)$ and the periodic wavelet expansion $f(x) = \sum_{k=0}^{2^{J_0}-1} c_k^{J_0} \tilde{\phi}_{J_0,k}(x) + \sum_{j=J_0}^{J-1} \sum_{k=0}^{2^j-1} d_k^j \tilde{\psi}_{j,k}(x)$. Where J_0 satisfy $0 \leq J_0 \leq J$ and the decay of the wavelet coefficient is given by the following theorem [Nilsen (1998)]:

Theorem
Let $P = D/2$ be the number of vanishing moments for a wavelet $\psi_{j,k}$ and let $f \in C^P(R)$. Then the wavelet coefficients decay as $|d_{j,k}| \leq C_P 2^{-j(P+\frac{1}{2})} \times max_{\xi \in I_{j,k}} |f^{(P)}(\xi)|$.

3. Wavelet-Taylor Galerkin Method (W-TGM)

In Wavelet-Taylor Galerkin method the time discretization is performed before the spatial discretization. This implies that the corrections due to Taylor series act at the level of the spatially continuous differential equation. Thus the present method is in contrast with other approaches wherein Taylor series or Pade rational expansions in time are applied to the spatially discretized equations. It is clear that, due to the presence of the consistent mass matrix (CMM) in the wavelet Galerkin equations for transient problems, the proposed method offers substantial advantages in the computational efficiency with respect to such Galerkin-Taylor alternatives. Consider the equation of the form

$$u_t = \mathcal{L}u + \mathcal{N}f(u) \tag{6}$$

with the initial condition

$$u(x,0) = u_0(x), \quad 0 \leq x \leq 1 \tag{7}$$

and with the suitable boundary condition. We explicitly separate evolution Eq. (6) into a linear part, $\mathcal{L}u$ and nonlinear part, $\mathcal{N}f(u)$, where the operation $\mathcal{L}$ and $\mathcal{N}$ are constant-coefficient differential operation that do not depend upon time t. The function $f(u)$ is nonlinear. A simple example of Eq. (6) is the classical diffusion (or heat) equation

$$u_t = \nu u_{xx}, \quad \nu > 0. \tag{8}$$

Remark. Although we do not address multidimensional problems in this paper, we note that the Navier-Stokes equations may also be written in the form (6). Consider

$$u_t + \frac{1}{2}[u \cdot \nabla u + \nabla(u \cdot u)] = \nu \nabla^2 u - \nabla p, \tag{9}$$

where

$$\operatorname{div} u = 0, \tag{10}$$

and p denotes the pressure. Applying divergence operator to both sides of (9) and using (10), we obtain

$$\Delta p = f(u), \tag{11}$$

where $f(u) = -\frac{1}{2}\nabla[u \cdot \nabla u + \nabla(u \cdot u)]$ is a nonlinear function of u. Equation (6) is formally obtained by setting

$$\mathcal{L}u = \nu \nabla^2 u \tag{12}$$

and

$$\mathcal{N}f(u) = \frac{1}{2}[u \cdot \nabla u + \nabla(u \cdot u)] - \nabla(\Delta^{-1} f(u)). \tag{13}$$

A one-dimensional model that may be thought of as a prototype for the Navier-Stokes equation is

$$u_t = \mathcal{H}(u)u. \tag{14}$$

where $\mathcal{H}(\cdot)$ is the Hilbert transform. The presence of the Hilbert transform in (14) introduces a long-range interaction which models that found in the Navier-Stokes equations.

3.1. *W-TGM for heat equation*

We consider now the periodic initial-value problem for the heat equation. For heat equation $\mathcal{L} = \nu \frac{\partial^2}{\partial_x^2}$, then equation becomes

$$\begin{aligned} u_t &= \nu u_{xx} + f(x), \;\; t > 0 \\ u(x,0) &= h(x), \qquad\quad 0 \le x \le 1 \\ u(x,t) &= u(x+1,t), \quad t \ge 0, \end{aligned} \tag{15}$$

where ν is a positive constant, $f(x) = f(x+1)$ and $h(x) = h(x+1)$.

3.1.1. *Scheme based on forward (or Euler) time stepping*

Let us first leave the spatial variable x continuous and discretize only the time to obtain the Euler scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} = \nu u_{xx}^n + f(x). \tag{16}$$

We are expressing the difference approximation to u_t at time level n by forward-time Taylor series expansion, including second and third time derivatives, which gives

$$(u_t)^n = \frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2} u_{tt}^n - \frac{\Delta t^2}{6} u_{ttt}^n - O(\Delta t^3). \tag{17}$$

3.1.2. *Scheme with a mixed temporal spatial correction (W-TGM)*

Now, successive differentiations of Eq. (15) indicate that

$$u_{tt} = \nu^2 u_{xxxx} + \nu f''(x) \quad \text{and} \quad u_{ttt} = \nu^2 (u_t)_{xxxx} \tag{18}$$

combining Eqs. (17) and (18), the semi-discrete equation (16) is replaced by the following equation:

$$\frac{u^{n+1} - u^n}{\Delta t} - \frac{\nu^2 \Delta t^2}{6}(u_t)^n_{xxxx} = \nu u^n_{xx} + \frac{\nu^2 \Delta t}{2} u^n_{xxxx} + \nu \frac{\Delta t}{2} f''(x) + f(x). \tag{19}$$

In Eq. (19), the term involving u_{xx} is not an additional artificial diffusion term, but part of the difference approximation to u^n_t. Whereas the adopted mixed form simply leads to modify the usual CMM. This is achieved by replacing u^n_t in equation (19) by $(u^{n+1} - u^n)/\Delta t$, so that after applying wavelet Galerkin method (19) transforms into the following schemes:

$$\left(I - \frac{\nu^2 \Delta t^2}{6} D^{(4)}\right)\left(c^{n+1}_u - c^n_u\right) = \nu \Delta t D^{(2)} c^n_u + \frac{\nu^2 \Delta t^2}{2} D^{(4)} c^n_u + \frac{\nu \Delta t^2}{2} d_{f''} + \Delta t d_f, \tag{20}$$

where c_u denote the vector of scaling function coefficients corresponding to u and $d_f, d_{f''}$ is given by denote the vector of scaling function coefficients corresponding to f and f''. We will refer to the matrix $D^{(d)}$ as the differentiation matrix of order d. Derivation of matrix D^d is given in [Nilsen (1998)].

3.1.3. *Scheme with spatial correction (W-TGMS)*

The term u_{tt} can also be expressed in a purely spatial form. By noticing that

$$u_{ttt} = \nu^3 u_{xxxxxx} + \nu^2 f''''(x), \tag{21}$$

the Taylor series expansion finally provides

$$\begin{aligned} \frac{u^{n+1} - u^n}{\Delta t} &= \nu u^n_{xx} + \frac{\nu^2 \Delta t}{2} u^n_{xxxx} + \frac{\nu^3 \Delta t^2}{6} u_{xxxxxx} \\ &\quad + \frac{\nu^2 \Delta t^2}{6} f'''' + \nu \frac{\Delta t}{2} f''(x) + f(x). \end{aligned} \tag{22}$$

The Galerkin discretization scheme gives

$$\begin{aligned} (c^{n+1}_u - c^n_u) &= \nu \Delta t D^{(2)} c^n_u + \frac{\nu^2 \Delta t^2}{2} D^{(4)} c^n_u + \frac{\nu^3 \Delta t^3}{6} D^{(6)} c^n_u \\ &\quad + \frac{\nu^2 \Delta t^3}{6} d_{f''''} + \frac{\nu \Delta t^2}{2} d_{f''} + \Delta t d_f. \end{aligned} \tag{23}$$

Remark 1.

Here it may noted that in the finite element context terms such as $(u_t)^n_x, (u_t)^n_{xx}$ etc. are left in the spatial-temporal form because the elimination of u_t through given PDE would introduce higher-order spatial derivatives. This would preclude the use of finite elements with C^0-continuity (Lagrange family). Also the Taylor Galerkin schemes are usually limited to second-order time derivatives for high order PDEs. Such difficulties are not encountered here in our W-TGM approach. However, in W-TGM specially while using schemes like W-TGMS one has to carefully choose D as per the basic results of [Eirola (1992)] to account for the differentiability of ϕ.

Remark 2.
When we are using W-TGM, the presence of a CMM is an explicit time stepping scheme represents a serious disadvantage from the point of view of the computational efficiency of the method. It has been also shown that it is possible to exploit the beneficial effect of the CMM in a lumped-explicit context. This may be achieved by using a two-pass explicit procedure, we will refer to this scheme as twoW-TGM. This twoW-TGM procedure for the W-TGM scheme (20) would read as:

$$\begin{aligned} (c_u^{n+1} - c_u^n)^{(1)} &= \nu \Delta t D^{(2)} c_u^n + \frac{\nu^2 \Delta t^2}{2} D^{(4)} c_u^n + \frac{\nu \Delta t^2}{2} d_{f''} + \Delta t d_f \\ (c_u^{n+1} - c_u^n)^{(2)} &= (c_u^{n+1} - c_u^n)^{(1)} + \frac{\nu^2 \Delta t^2}{6} D^{(4)} (c_u^{n+1} - c_u^n)^{(1)}. \end{aligned} \tag{24}$$

3.1.4. *Scheme based on leap-frog time stepping*

Now replace the time derivative with the standard leap-frog discretization. This gives

$$\frac{u^{n+1} - u^{n-1}}{2\Delta t} = \nu u_{xx}^n + f(x). \tag{25}$$

The use of centered difference to discretize the right-hand side of Eq. (25) produces a method which is second-order accurate in space and time. To obtain an improved order of accuracy in Δt we shall apply a Taylor-Galerkin method based on the following forward and backward time expansions.

$$\begin{aligned} u^{n+1} &= u^n + \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n + \frac{\Delta t^3}{6} u_{ttt}^n + O(\Delta t^4) \\ u^{n-1} &= u^n - \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n - \frac{\Delta t^3}{6} u_{ttt}^n + O(\Delta t^4) \end{aligned} \tag{26}$$

from which we deduce an improved approximation to the temporal derivative at time $t = n\Delta t$ in the form

$$u_t^n = \frac{u^{n+1} - u^{n-1}}{2\Delta t} - \frac{\Delta t^2}{6} u_{ttt}^n. \tag{27}$$

Now with this improved approximation in to left-hand side of Eq. (25), followed by substitution of the third time-derivative with the mixed form in Eq. (27), the associated wavelet-Galerkin equations are

$$\left(I - \frac{\nu^2 \Delta t^2}{6} D^{(4)} \right) \left(c_u^{n+1} - c_u^{n-1} \right) = 2\nu \Delta t D^{(2)} c_u^n + 2\Delta t d_f. \tag{28}$$

Here W-TGMS scheme becomes

$$\left(c_u^{n+1} - c_u^{n-1}\right) = 2\nu\Delta t D^{(2)} c_u^n + \frac{\nu^3\Delta t^3}{3} D^{(6)} c_u^n + \frac{\nu^2\Delta t^3}{3} d_{f''''} + 2\Delta t d_f. \tag{29}$$

Here twoW-TGM procedure for the W-TGM scheme (28) would read as:

$$\begin{aligned} (c_u^{n+1} - c_u^{n-1})^{(1)} &= 2\nu\Delta t D^{(2)} c_u^n + 2\Delta t d_f \\ (c_u^{n+1} - c_u^{n-1})^{(2)} &= (c_u^{n+1} - c_u^{n-1})^{(1)} + \frac{\nu^2\Delta t^2}{6} D^{(4)} (c_u^{n+1} - c_u^{n-1})^{(1)}. \end{aligned} \tag{30}$$

3.1.5. *Scheme based on Crank-Nicolson (CN) time stepping*

The standard Crank-Nicolson discretization gives

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}\nu(u_{xx}^n + u_{xx}^{n+1}) + f(x). \tag{31}$$

If a central difference is employed to discretize the spatial term, the method is unconditionally stable and second-order accurate in space and time. To obtain an improved order of accuracy in Δt we shall apply a Taylor-Galerkin method based on the following Taylor series expansions:

$$\begin{aligned} u^{n+1} &= u^n + \Delta t u_t^n + \frac{\Delta t^2}{2} u_{tt}^n + \frac{\Delta t^3}{6} u_{ttt}^n + \cdots \\ u^n &= u^{n+1} - \Delta t u_t^{n+1} + \frac{\Delta t^2}{2} u_{tt}^{n+1} - \frac{\Delta t^3}{6} u_{ttt}^{n+1} + \cdots \end{aligned} \tag{32}$$

Combination of these two gives

$$\frac{u^{n+1} - u^n}{\Delta t} = \frac{1}{2}(u_t^n + u_t^{n+1}) + \frac{\Delta t}{4}(u_{tt}^n - u_{tt}^{n+1}) + \frac{\Delta t^2}{12}(u_{ttt}^n + u_{ttt}^{n+1}) \tag{33}$$

replacing the time derivatives by spatial derivatives, the associated wavelet Taylor-Galerkin equations based on CN time stepping scheme are

$$\begin{aligned} \left(I - \frac{\nu^2\Delta t^2}{6} D^{(4)}\right)\left(c_u^{n+1} - c_u^n\right) &= \frac{\nu\Delta t}{2} D^{(2)}\left(c_u^n + c_u^{n+1}\right) \\ &\quad + \frac{\nu^2\Delta t^2}{4} D^{(4)}\left(c_u^n - c_u^{n+1}\right) + \Delta t d_f. \end{aligned} \tag{34}$$

Here W-TGMS scheme becomes

$$\begin{aligned} \left(c_u^{n+1} - c_u^n\right) &= \frac{\nu\Delta t}{2} D^{(2)}(c_u^n + c_u^{n+1}) + \frac{\nu^2\Delta t^2}{4} D^{(4)}(c_u^n - c_u^{n+1}) \\ &\quad + \frac{\nu^3\Delta t^3}{12} D^{(6)}\left(c_u^n + c_u^{n+1}\right) + \frac{\nu^2\Delta t^3}{6} d_{f''''} + \Delta t d_f. \end{aligned} \tag{35}$$

3.2. *W-TGM for inviscid Burgers equation (quasilinear hyperbolic conservation equation)*

For inviscid Burgers equation $\mathcal{N}f(u) = u\frac{\partial u}{\partial x}$, then equation becomes

$$u_t + uu_x = 0. \tag{36}$$

A second-order Taylor-Galerkin method is used to obtain a discrete approximation of (36). The equation is first discretized in time by considering a Taylor series expansion in the time step $\Delta t = t^{n+1} - t^n$ up to second order

$$(u_t^n) = \frac{u^{n+1} - u^n}{\Delta t} - \frac{\Delta t}{2}u_{tt}^n - O(\Delta t^2). \tag{37}$$

The first-order and second-order time derivative in (37) are then expressed from the governing Eq. (36) in the form

$$\begin{aligned} u_t &= -uu_x \\ u_{tt} &= -u_t u_x - uu_{xt}. \end{aligned} \tag{38}$$

Now with this improved approximation in to left side of Eq. (36), followed by substitution of the second time-derivative with the mixed form in Eq. (37), the semi discrete equation is

$$\frac{u^{n+1} - u^n}{\Delta t} = -u^n u_x^n + \frac{\Delta t}{2}\left[-\left(\frac{u^{n+1} - u^n}{\Delta t}\right)u_x - u\partial_x\left(\frac{u^{n+1} - u^n}{\Delta t}\right)\right], \tag{39}$$

where spatial discretization is done by WGM.

4. Results of Numerical Experiments

The Lax-Wendroff spatial discretization will be performed on a spatial grid. Let $x_i = i\Delta x$, where $\Delta x = 1/N$ and $i = 0, 1, \ldots, N-1$, $N = 2^j$.

4.1. *Accuracy results*

The error produced by the numerical schemes was measured against the values of the analytical solution u_e by the L_∞ norm calculated as

$$\|u\|_{L_\infty} = \max_{k=0,1,\ldots,2^j-1} |u(k/(2^j))|.$$

4.2. *Heat equation*

We have tested W-TGM on heat equation with different set of initial condition and different value of function $f(x)$.

Case 1: $h(x) = 0, f(x) = \sin(2\pi x)$

In Tables 1(a–c) errors in L_∞ norm of the solution obtained by W-TGM scheme, W-TGMS and twoW-TGM are compared with those resulting from Lax-Wendroff scheme and WGM. Results in Table 1(a) correspond to the computations with

Table 1. Accuracy of results given by the L_∞ norm using Euler time stepping scheme.

(a)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM	W-TGMS
4	0.0005(0.01)	3.178×10^{-6}	3.138×10^{-7}	1.3113×10^{-11}	1.457×10^{-11}	1.938×10^{-11}
6	0.0005(0.01)	3.142×10^{-6}	3.138×10^{-7}	3.816×10^{-17}	2.498×10^{-16}	6.247×10^{-12}
7	0.0005(0.01)	3.140×10^{-6}	3.138×10^{-7}	1.509×10^{-16}	1.2663×10^{-16}	6.247×10^{-12}

(b)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM
4	0.005(0.1)	3.161×10^{-4}	2.907×10^{-5}	1.99×10^{-6}	1.99×10^{-6}
6	0.005(0.1)	3.125×10^{-4}	3.103×10^{-5}	9.709×10^{-9}	9.709×10^{-9}
7	0.005(0.1)	3.123×10^{-4}	3.103×10^{-5}	9.431×10^{-10}	9.304×10^{-10}

(c)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM
4	0.05(1)	0.03	0.0026	2.073×10^{-4}	2.073×10^{-4}
6	0.05(1)	0.0297	—	3.04×10^{-5}	2.8×10^{-6}
7	0.05(1)	—	—	2.53×10^{-5}	—

$D = 10$ and those in Tables 1(b) and 1(c) correspond to $D = 6$. In the table the notation $.x(.y)$ would mean that the target time '$.y$' is reached by time marching with step size of '$.x$'. Further a '—' in the table denotes that the scheme is unstable. For Euler time stepping our schemes exhibit higher accuracy in comparison to other methods. It is also to be noted from Table 1(c) that while W-TGM permit larger time stepping, Lax-Wendroff and WGM are seen to get unstable.

Tables 2(a–c) depict that under leap-frog time stepping to our wavelet-Taylor Galerkin schemes are superior to WGM and Lax-Wendroff methods. Also it is to be noted that under leap-frog time stepping W-TGMS approach gets as accurate as W-TGM and twoW-TGM. Here, results in Table 2(a) correspond to the computations with $D = 10$ and those in Tables 2(b) and 2(c) correspond to computations with $D = 6$.

The accuracy of the W-TGM and W-TGMS based on CN time stepping are compared in Tables 3(a–c) with WGM and Lax-Wendroff method. Note that our schemes under CN time marching strategy will again lead to better accuracy in numerical solution.

Case 2: Let us consider $f(x) = 0$ and following initial condition

$$h(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x \leq 1 \end{cases} \tag{40}$$

that has a discontinuous derivative at $x = \frac{1}{2}$. In Figs. 1(a–b) solution to the problem for $\Delta t = 0.0005$, $\Delta x = 2^{-4}$ and $\nu = 1$ by FD and Lax-Wendroff methods are

Table 2. Accuracy of results given by the L_∞ norm using leap-frog time stepping scheme.

(a)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM	W-TGMS
4	0.0005(0.01)	8.026×10^{-8}	4.262×10^{-11}	1.628×10^{-11}	1.628×10^{-11}	1.631×10^{-11}
6	0.0005(0.01)	5.031×10^{-9}	2.631×10^{-11}	2.897×10^{-16}	2.862×10^{-16}	2.949×10^{-16}
7	0.0005(0.01)	1.251×10^{-9}	2.631×10^{-11}	2.949×10^{-17}	2.949×10^{-17}	2.949×10^{-17}

(b)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM
4	0.005(0.1)	7.957×10^{-5}	2.093×10^{-6}	2.067×10^{-6}	2.067×10^{-6}
6	0.005(0.1)	4.905×10^{-7}	3.467×10^{-8}	8.487×10^{-9}	8.486×10^{-9}
7	0.005(0.1)	1.155×10^{-7}	$-(0.0093)$	$-(6.81 \times 10^{-4})$	2.143×10^{-4}

(c)

j	Δt	Lax-Wendroff	WGM	W-TGM	twoW-TGM
4	0.05(1)	7.3×10^{-4}	2.174×10^{-4}	1.926×10^{-4}	1.926×10^{-4}
6	0.05(1)	—	—	$-(1.30)$	8.141×10^{-4}
7	0.05(1)	—	—	—	—

Table 3. Accuracy of results given by the L_∞ norm using Crank-Nicolson time stepping scheme.

(a)

j	Δt	Lax-Wendroff	WGM	W-TGM	W-TGMS
4	0.0005(0.01)	8.026×10^{-8}	1.632×10^{-11}	2.2903×10^{-11}	2.293×10^{-11}
6	0.0005(0.01)	5.0342×10^{-9}	3.286×10^{-12}	3.139×10^{-16}	6.576×10^{-12}
7	0.0005(0.01)	1.254×10^{-9}	3.286×10^{-12}	2.776×10^{-17}	6.576×10^{-12}

(b)

j	Δt	Lax-Wendroff	WGM	W-TGM
4	0.005(0.1)	7.96×10^{-6}	2.063×10^{-6}	2.067×10^{-6}
6	0.005(0.1)	4.938×10^{-7}	5.179×10^{-9}	8.31×10^{-9}
7	0.005(0.1)	1.186×10^{-7}	2.721×10^{-9}	5.207×10^{-10}

(c)

j	Δt	Lax-Wendroff	WGM	W-TGM
4	0.05(1)	7.338×10^{-4}	1.889×10^{-4}	1.975×10^{-4}
6	0.05(1)	4.062×10^{-5}	2.120×10^{-6}	7.563×10^{-7}
7	0.05(1)	5.806×10^{-6}	2.852×10^{-6}	4.365×10^{-8}

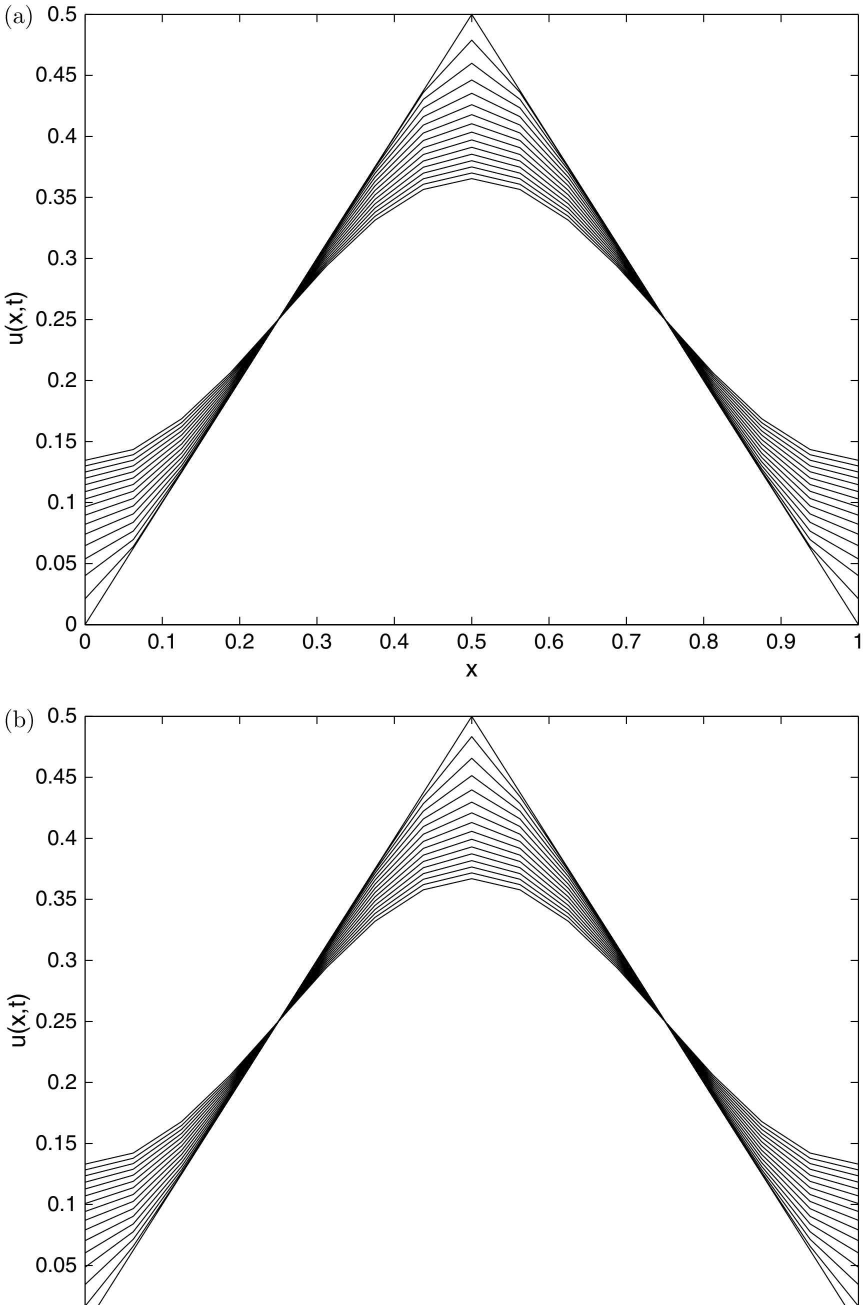


Fig. 1. Solution of heat equation (a) FD (b) Lax-Wendroff method with $\Delta t = 0.0005$, $\Delta x = 2^{-4}$ and $\nu = 1$ at $t = 0.05$.

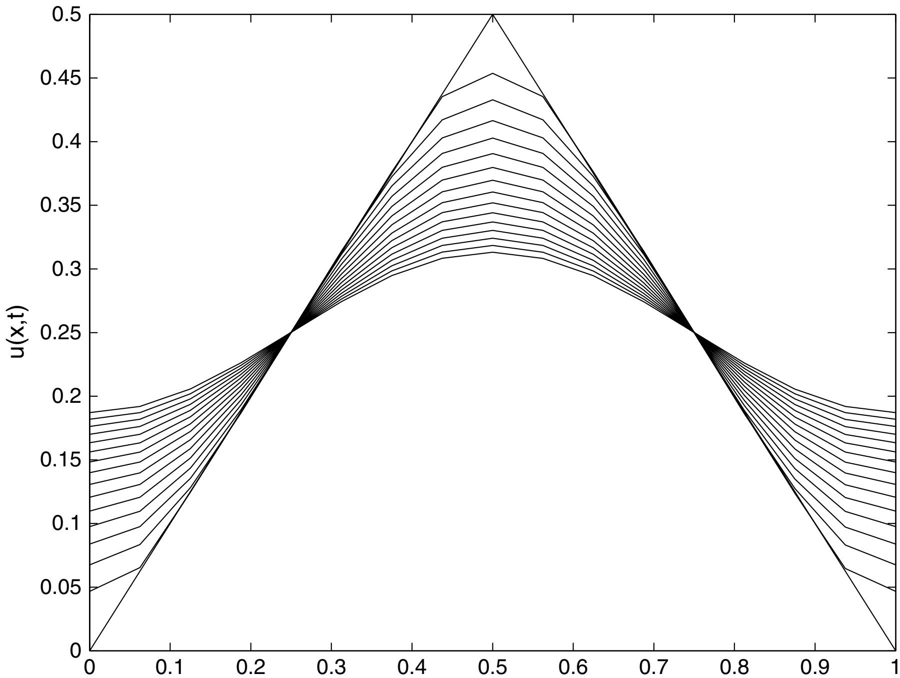


Fig. 2. Solution of heat equation using WGM based on Euler time stepping scheme with $\Delta t = 0.0005, J = 4$ and $\nu = 1$ at $t = 0.05$.

presented. Figures 2 and 3 represent the corresponding solution to the problem as obtained by WGM and W-TGM. Here one can note that the solution due to our W-TGM gets smooth quicker indicating a faster decay of sharp peak in the initial condition.

Figure 4 illustrates the evolution of (40) by FD based Crank-Nicolson scheme for $\Delta t = 0.0005$, $\Delta x = 10^{-6}$ and $\nu = 1$ and the slow decay of high frequency components of the initial condition. We have implemented our W-TGM scheme under CN time stepping for the same problem and display the result in Fig. 5 for $J = 6$. Here one can note that there is a proper decay of the sharp peak in the initial condition.

4.3. *Convection equation*

Consider the scalar convection equation in one dimension. For convection equation $\mathcal{L} = a\frac{\partial}{\partial_x}$, then equation becomes

$$u_t = au_x, \tag{41}$$

where a is positive constant. The accuracy of the proposed W-TGM for hyperbolic problems has been verified numerically first on the classical test problem of convection of Gaussian profile.

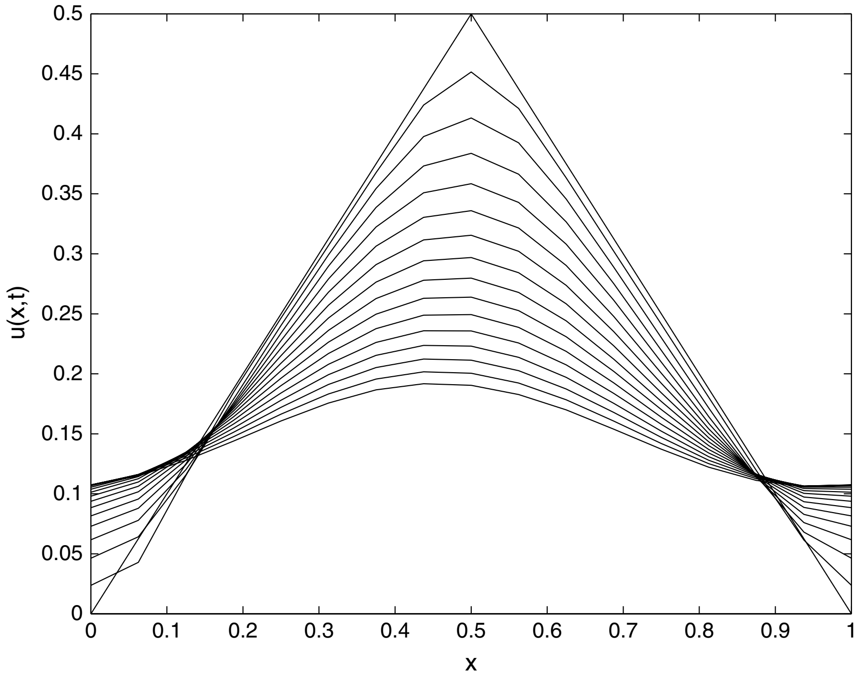


Fig. 3. Solution of heat equation using W-TGM based on Euler time stepping scheme with $\Delta t = 0.0005$, $J = 4$ and $\nu = 1$ at $t = 0.05$.

We assume that the solution is periodic of some large period for instance say, four. A comparison of the solution obtained by WGM and W-TGM with the exact solution is made in Figs. 6(a–b) respectively illustrates the relative superiority of W-TGM with hyperbolic equation. In addition L_∞-norm of the errors in solution due to W-TGM, twoW-TGM and WGM under Euler and leap-frog time stepping schemes shown in Tables 4(a–b) depict the stability and accuracy of our schemes.

4.4. *Inviscid Burgers' equation*

Here we are showing numerical results for the inviscid Burgers' equation with two different set of initial condition to illustrate the applicability of proposed W-TGM scheme in nonlinear context.

Case 1:
In this case we compute the solution using initial condition

$$u(x, 0) = \sin(2\pi x). \tag{42}$$

In this case to begin with we find the solution of undamped wave equation with initial condition (42) and imposed periodic boundary conditions. In Figs. 7(a–b) we present the solutions due to FD scheme and W-TGM under Euler time stepping

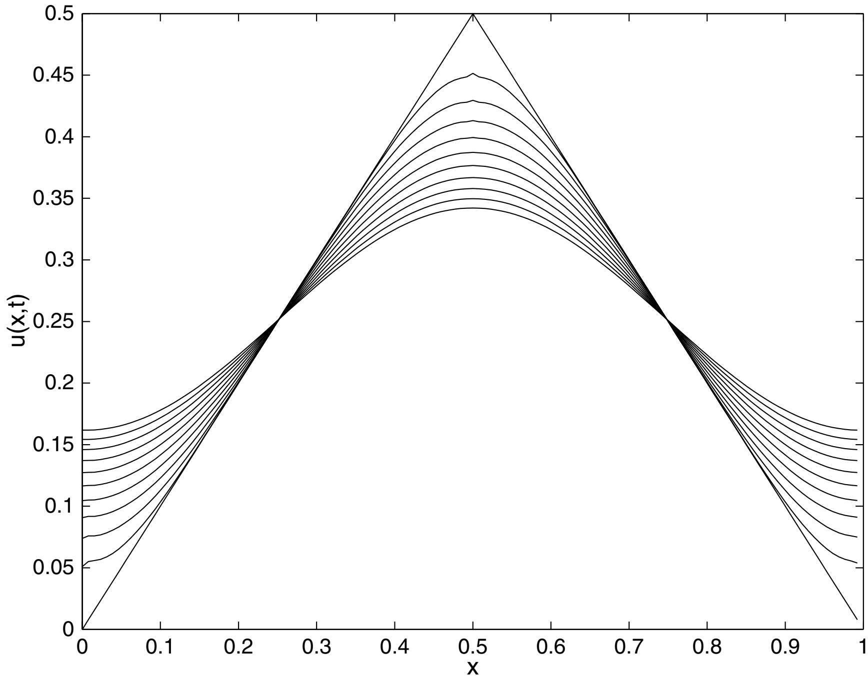


Fig. 4. Solution of heat equation using finite difference based on CN time stepping scheme with $\Delta t = 0.0005, \Delta x = 2^{-6}$ and $\nu = 1$ at $t = 0.05$.

scheme. While solution due to FD scheme develops local oscillations, the solution due to W-TGM continues to be smooth.

Case 2:
Here we are using initial condition

$$u(x,0) = \begin{cases} 1 + x & -1 \leq x \leq 0 \\ 1 - x & 0 \leq x \leq 1. \end{cases} \tag{43}$$

We determine solution of undamped wave equation with an initial triangular wave (43) and imposed periodic boundary conditions. The exact solution of this problem in the domain $-1 \leq x \leq 1$ with $u(-1,t) = 0$ predicts an infinitely steep gradient at $x = 1$ at time $t = 1$ and the area under the initial curve remains constant for all time. The numerical solution by W-TGM under Euler time stepping scheme presented in Fig. 8(a) agrees with the exact solution until $t = 0.90$ when mild oscillations appear in solution indicating the presence of steep gradient (shock) as predicted by the exact solution at $t = 1$. However, it is to be noted that these oscillations are confined to the vicinity of the shock and not felt in other parts of the domain. It is known [Restrepo and Leaf (1995)] that the Fourier bases produce oscillations which are present throughout the domain. In Fig. 8 (b) solution to the problem by

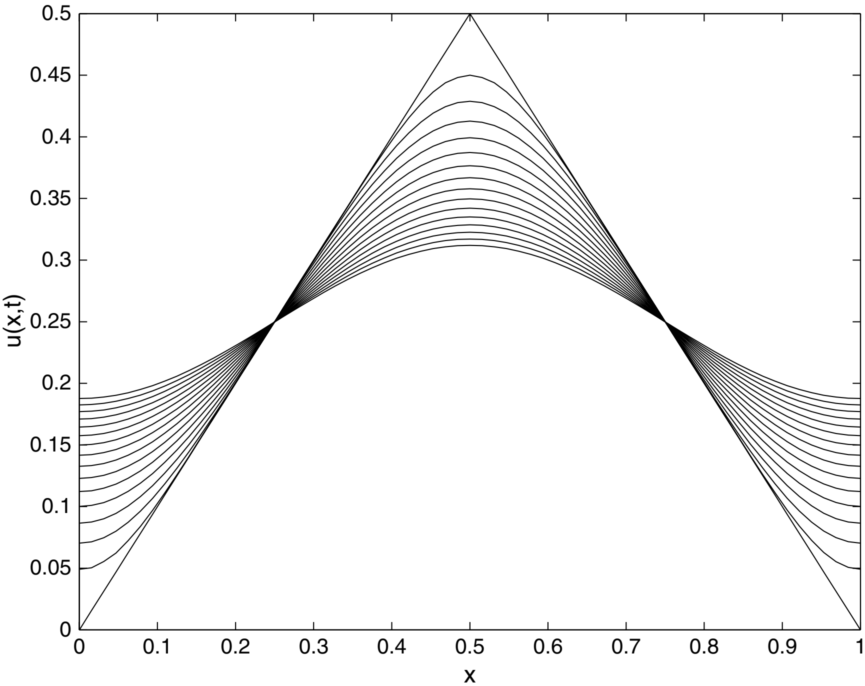


Fig. 5. Solution of heat equation using W-TGM based on CN time stepping scheme with $\Delta t = 0.0005$, $j = 6$ and $\nu = 1$ at $t = 0.05$.

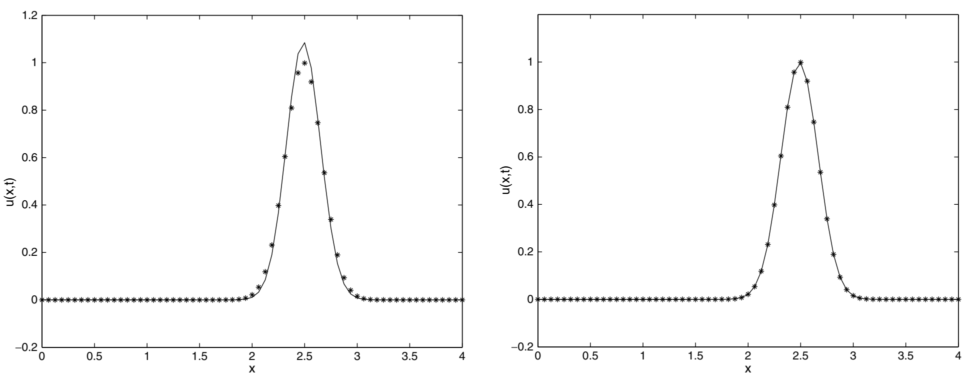


Fig. 6. Solution of convection equation based on Euler time stepping scheme (a) WGM scheme (b) W-TGM scheme $(-, \text{exact}, \ldots, \Delta t = 0.01)$.

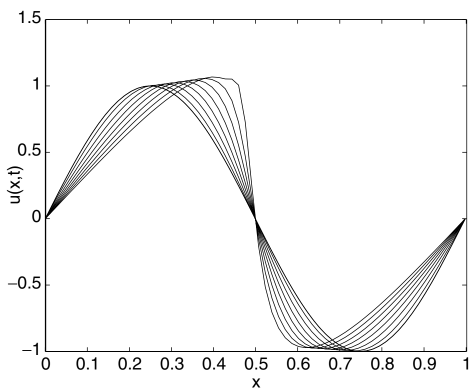
Table 4. Accuracy of results given by the L_∞ norm.

(a) (using Euler time stepping)

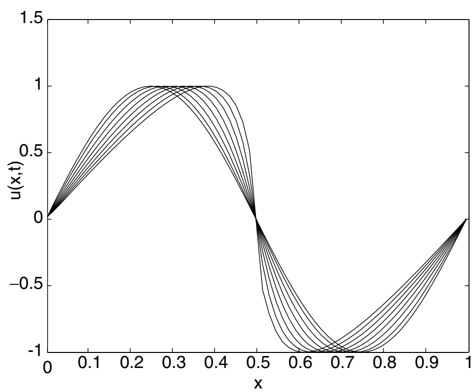
j	Δt	WGM	W-TGM	twoW-TGM
4	0.01(0.5)	0.0861	0.0023	0.0023
6	0.01(0.5)	—	5.227×10^{-5}	5.242×10^{-5}

(b) (using leap-frog time stepping)

j	Δt	WGM	W-TGM	twoW-TGM
4	0.01(0.5)	0.0214	0.0210	0.0210
6	0.001(0.05)	0.0021	0.0021	0.0021
6	0.0001(0.05)	2.101×10^{-4}	2.101×10^{-4}	2.101×10^{-4}



(a)



(b)

Fig. 7. Solutions of inviscid Burgers' equation at various time step with $\Delta t = 0.005$. Left: By finite difference method taking $\Delta x = 2^6$. Right: By W-TGM scheme taking $j = 6$.

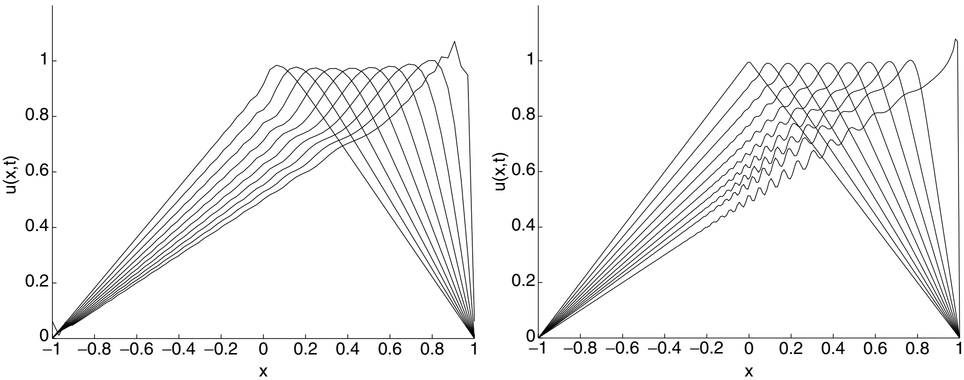
Fourier-Galerkin method is provided. Here the occurrence of a steep gradient gives rise to well marked global oscillations around $t = 0.7$.

5. Theoretical Stability of the Linearized Schemes

We use the notion of asymptotic stability of a numerical method as it is defined in [Canuto *et al.* (1988)] for a discrete problem of the form

$$\frac{dU}{dt} = LU,$$

where L is assumed to be a diagonalizable matrix.



(a) (b)

Fig. 8. Left: Local oscillation near shock. Right: Global oscillation in Fourier-Galerkin solution.

Definition
The region of absolute stability of a numerical method is defined for the scalar model problem

$$\frac{dU}{dt} = \lambda U$$

to be the set of all $\lambda \Delta t$ such that $\|U^n\|$ is bounded as $t \to \infty$. Finally we say that a numerical method is asymptotically stable for a particular problem if, for sufficiently small $\Delta t > 0$, the product of Δt times every eigenvalue of L lies within the region of absolute stability.

Forward (Euler) scheme:
The region of absolute stability for this scheme is the circle of radius 1 and center $(-1,0)$.

Leap-Frog scheme:
The stability condition for this scheme is that $\lambda \Delta t$ be on the imaginary axis and that $|\lambda \Delta t| \leq 1$.

Crank-Nicolson scheme:
This method is absolute stable in entire left-half plane.

5.1. *Stability for heat equation*

We consider the heat equation

$$u_t = \nu u_{xx}.$$

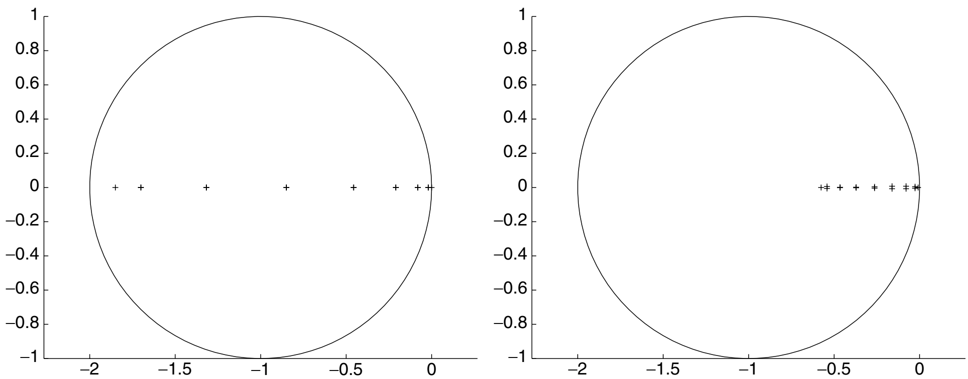


Fig. 9. Absolute stability region for forward Euler and Δt times the eigenvalues of L_4 for Daubechies scaling functions, where $\Delta t = 0.0005$, $\nu = 1$, left for $D = 6$, right for $D = 16$.

For W-TGM based on Euler time stepping L_j be the matrix defined as: $L_j = A_j^{-1} B_j$ where $A_j = I - \frac{\nu^2 \Delta t^2}{6} D^{(4)}, B_j = \nu D^{(2)} + \frac{\nu^2 \Delta t}{2} D^{(4)}$, so that the discretized Heat equation becomes

$$\frac{dc_u}{dt} = L_j c_u.$$

Figure 9 shows the absolute stability region based on Euler time stepping together with the position of the $\Delta t \lambda_L(i)$ for Daubechies scaling functions, with some Δt which should satisfy the stability condition. The region of absolute stability for W-TGM based on CN time stepping scheme are plotted in Fig. 10.

5.2. *Stability for convection equation*

$$u_t = a u_x.$$

Here for W-TGM based on Euler time stepping L_j is defined by $L_j = A_j^{-1} B_j$, where $A_j = I - \frac{a^2 \Delta t^2}{6} D^{(2)}, B_j = a D^{(1)} + \frac{a^2 \Delta t}{2} D^{(2)}$. Table 5 shows the asymptotic stability condition for $j = 4, 6$ in an Euler scheme.

Figures 11 and 12 shows the stability region of W-TGM based on Euler and leap-frog time stepping scheme.

5.3. *Stability for inviscid linearized Burgers' equation*

We consider the linearized inviscid Burgers equation

$$u_t + \alpha u_x = 0,$$

where the linearization coefficient α stands for the value of u. Because at the initial time $u_0(x) = \sin(2\pi x)$, and because the amplitude of u decreases with time, we

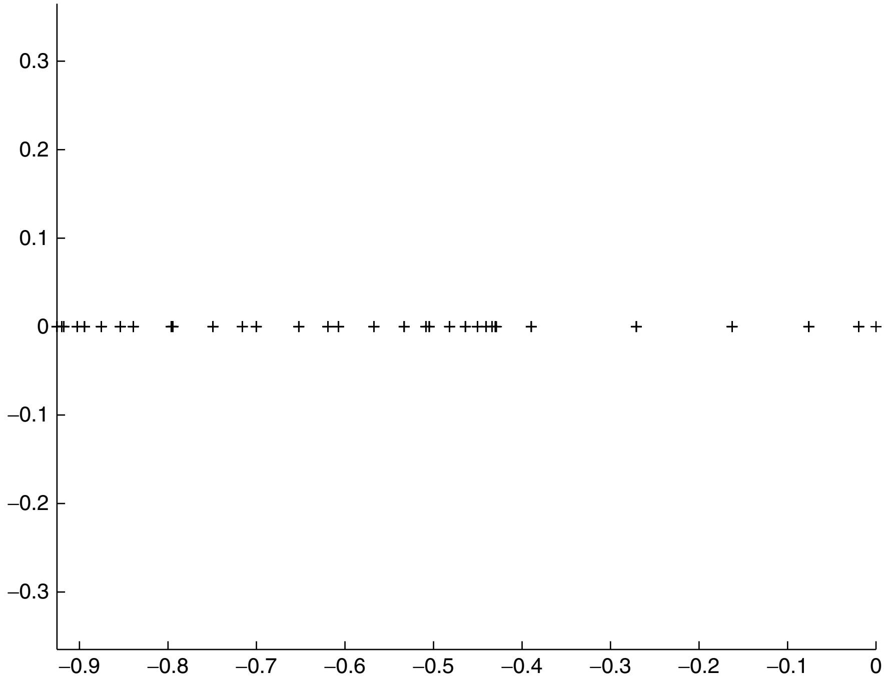


Fig. 10. For Crank-Nicolson scheme Δt times the eigenvalues of L_6 for Daubechies scaling functions, $\Delta t = 0.0005$.

Table 5.

j	No. of nodes	Stab. cond.
4	16	$\Delta t \leq 0.11$
6	64	$\Delta t \leq 0.01$

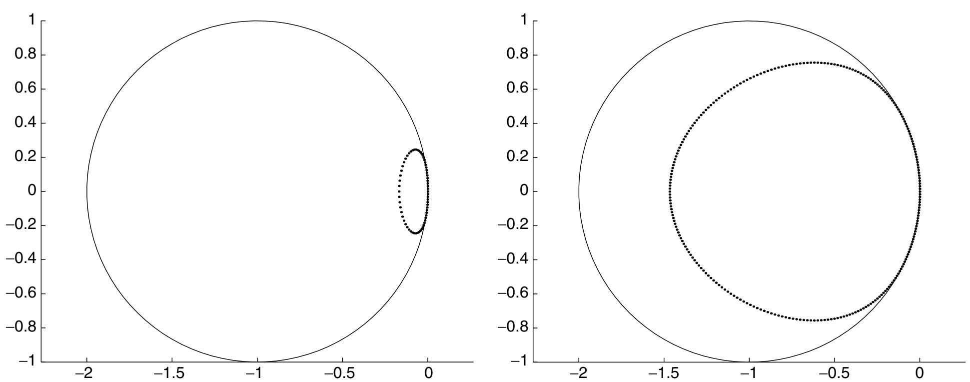


Fig. 11. Absolute stability region for forward Euler and Δt times the eigenvalues of L_j for Daubechies scaling functions, where $\Delta t = 0.01$, left for $j = 4$, right for $j = 6$.

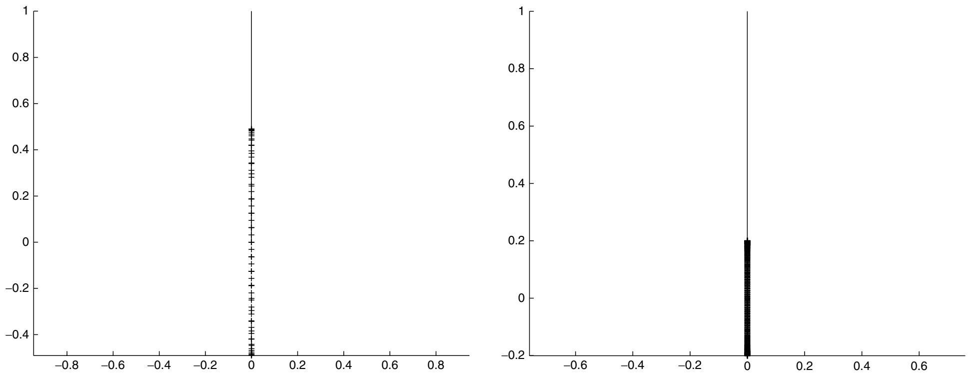


Fig. 12. Absolute stability region for leap-frog and Δt times the eigenvalues of L_j for Daubechies scaling functions, left for $\Delta t = 0.01, j = 4$ and right for $\Delta t = 0.001, j = 6$.

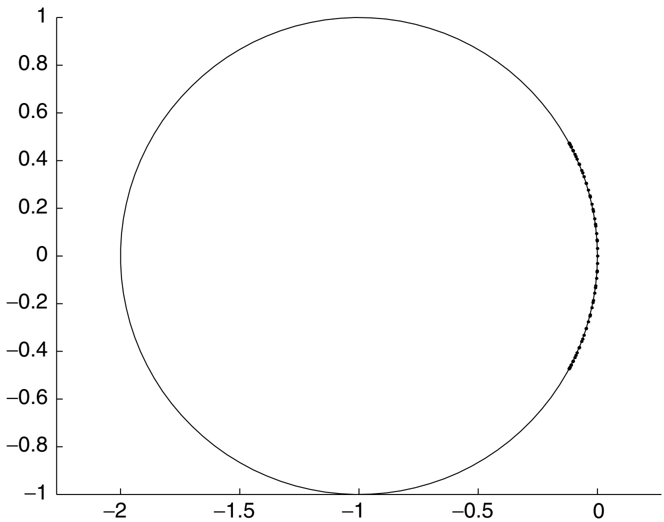


Fig. 13. Absolute stability region for Euler time stepping and Δt times the eigenvalues of L_j for Daubechies scaling functions, for $\Delta t = 0.005, j = 6$.

assume throughout that $|\alpha| \leq 1$. Here also $L_j = {A_j}^{-1} B_j$, where $A_j = I + \frac{\alpha \Delta t}{2} D^{(1)}$, $B_j = -\alpha D^{(1)}$, so that the discretized linearized Burgers' equation becomes

$$\frac{dc_u}{dt} = L_j c_u.$$

Figure 13 shows the absolute stability region for this scheme.

6. Conclusion

A set of stable and efficient numerical schemes called wavelet-Taylor Galerkin methods namely W-TGM, W-TGMS and twoW-TGM for parabolic and hyperbolic PDEs are introduced. The precedence of time discretization to space discretization and the use of Taylor-series up to third order in approximating time derivatives in conjunction with wavelet bases for expressing spatial terms renders robustness to the proposed schemes and makes them space and time accurate. Under Euler, leap-frog, CN time marching approaches our schemes W-TGM, twoW-TGM and W-TGMS are superior to WGM and Lax-Wendroff methods for parabolic equation i.e. heat transfer equation. In several instances while WGM and Lax-Wendroff get unstable, our methods remain stable and robust. With hyperbolic equations like convection equation and inviscid Burgers' equation W-TGM leads to superior and stable solution in comparison to FD, Lax-Wendroff, WGM etc. In dealing with unperturbed wave equation in $-1 \leq x \leq 1$ whose solution is known to develop steep gradient (or shock) at $x = 1$, while conventional Fourier Galerkin method leads to global oscillations in the solution, W-TGM gives raise to stable solution with mild oscillations highly localized to shock zone. In effect, we find that our wavelet-Taylor Galerkin methods lead to a stable and space time accurate solutions. Though the current implementations deal with one dimensional problems, the proposed methods are directly extendable to higher dimensions including to non-linear problems such as Navier-stokes equations. Further the methods are highly amendable to parallelization. Currently work is in progress in these two directions.

References

Amaratunga, K., Williams, J., Qian, S. and Weiss, J. (1994). Wavelet Galerkin solutions for 1D partial differential equations. *Int. J. Numer. Math. Eng.*, **37**: 2703–2716.

Beylkin, G., Coifman, R. and Rokhlin, V. (1991). Fast wavelet transforms and numerical algorithms. *Comm. Pure Appl. Math., XLIV*, pp. 141–183.

Canuto, C., Hussaini, M. Y., Quarteroni, A. and Zang, T. A. (1988). *Spectral Methods in Fluid Dynamics*, Springer-Verlag.

Daubechies, I. (1988). Orthonormal bases of compactly supported wavelets, *Comm. Pure Appl. Math.*, **41**: 906–966.

Donea, J. (1984). A Taylor-Galerkin method for convective transport problems. *Int. J. Numer. Methods Eng.*, **20**: 101–119.

Donea, J., Quartapelle, L. and Selmin, V. (1987). An analysis of time discretization in finite element solution of hyperbolic problems, *J. Comput. Phys.* **70**: 463.

Eirola, T. (1992). Sobolev characterization of solutions of dilation equations. *SIAM J. Math. Anal.*, **23**(4): 1015–1030.

Fatkulin, I. and Hesthaven, J. S. (2001). Adaptive high-order finite-difference method for nonlinear wave problems. *Journal of Sci. Comp.*, **16**(1).

Frohlich, J. and Schneider, K. (1997). An adaptive-vaguelette algorithm for the solution of PDEs. *J. Comp. Phys.*, **130**: 174–190.

Gazdag, J. (1973). Numerical convective schemes based on accurate computation of space derivatives. *J. Comput. Phys.*, **13**: 100–113.

Glowinski, R., Lawton, W., Ravachol, M. and Tenenbaum, E. (1990). Wavelet solutions of linear and non-linear elliptic, parabolic and hyperbolic problems in 1D. *Computing Methods in Applied Sciences and Engineering, SIAM, PA*, Chap. 4, pp. 55–120.

Holmstrom, M. (1999). Solving hyperbolic PDEs using interpolating wavelets. *SIAM J. Sci. Comput.*, **21**(2): 405–420.

Holmstrom, M. and Walden, J. (1998). Adaptive wavelets methods for hyperbolic PDEs. *Journal of Scientific Computing*, **13**(1): 19, 49.

Jameson, L. (1998). A wavelet-optimized very high order adaptive grid and order numerical method. *SIAM J. Sci. Comp.*, **19**: 1980–2013.

Kumar, B. V. R. and Mehra, M. (2005). Time accurate solution of Korteweg-de Vries equation using wavelet Galerkin method. *Applied Mathematics and Computation* **45**(5): 447–460.

Lax, P. D. and Wendroff, B. (1960). Systems of conservation laws. *Comm. Pure Appl. Math.*, **13**: 217.

Lax, P. D. and Wendroff, B. (1962). On the stability of difference schemes. *Comm. Pure Appl. Math.* **15**: 363.

Lax, P. D. and Wendroff, B. (1964). Difference schemes for hyperbolic equations with high order of accuracy. *Comm. Pure Appl. Math.*, **17**: 381.

Liandrat, J., Perrier, V. and Tchmitchian, Ph. (1992). Numerical resolution of nonlinear partial differential equations using wavelet approach. *Wavelets and Applications*, Boston, MA, pp. 227–238.

Meyer, Y. (1990). *Ondelettes et Operateurs, volume I: Ondelettes*, Hermann, Paris.

Nilsen, O. M. (1998). Wavelets in scientific computing. *PhD Thesis*, Technical University of Denmark, Lyngby.

Qian, S. and Weiss, J. (1992). Wavelets and the numerical solution of partial differential equations. *J. Comp. Phys.*

Restrepo, J. M. and Leaf, G. K. (1995). Wavelet Galerkin discretization of hyperbolic equations. *J. Comput. Phys.*, **122**: 118–128.

Vasilyev, O. V. and Bowman, C. (2000). Second generation wavelet collocation method for the solution of partial differential equations. *J. Comp. Phys.*, **165**: 660–293.

Vasilyev, O. V. (2003). Solving multi-dimensional evolution problems with localized structures using second generation wavelets, *Int. J. Comp. Fluid Dynamics*, **17**(2): 151–168.